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Ideals of rings of differential operators on algebraic curves (with an appendix by George Wilson[1])

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ABSTRACT

Let X be a smooth affine irreducible curve over $\mathbb{C}$ and let $\mathcal{D} = \mathcal{D}(X)$ be the ring of global differential operators on X. In this paper, we give a geometric classification of left ideals in $\mathcal{D}$ and study the natural action of the Picard group of $\mathcal{D}$ on the space of isomorphism classes of such ideals. Our results generalize the classification of left ideals of the first Weyl algebra $A_1(\mathbb{C})$ given in Berest and Wilson (2000, 2002) [15,16].

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1. Introduction

Let X be a smooth affine irreducible curve over $\mathbb{C}$ and let $\mathcal{D} = \mathcal{D}(X)$ be the ring of global differential operators on X. In this paper, we give a geometric classification of left ideals in $\mathcal{D}$ and study the natural action of the Picard group of $\mathcal{D}$ on the space of isomorphism classes of such ideals. Our results generalize the classification of left ideals of the first Weyl algebra $A_1(\mathbb{C})$, which is the ring of differential operators on the affine line $\mathbb{A}^1$ (see [15] and [16]); however, our methods are quite different from the methods used in those papers.

As shown in [15,16], the ideal classes of $A_1(\mathbb{C})$ are parametrized by finite-dimensional algebraic varieties $\mathcal{C}_n$ called the Calogero–Moser spaces. The starting point for the present paper was the observation of Crawley-Boevey (see [26]) that the same varieties $\mathcal{C}_n$ parametrize finite-dimensional irreducible representations of certain (infinite-dimensional) algebras associated to graphs. Specifically, the algebras in question are deformed preprojective algebras $\Pi^{\lambda}(Q)$ (see [29]); the corresponding graph Q is the framed Dynkin diagram of simplest type $\tilde{A}_0$. Trying to understand the relation between the ideals of $A_1(\mathbb{C})$ and irreducible representations of $\Pi^{\lambda}(Q)$, we came up with a new construction of the Calogero–Moser correspondence which, besides the Weyl algebra, applied to noncommutative deformations of Kleinian singularities corresponding to Dynkin diagrams of other types (see [13]). In this paper, we develop a geometric version of this construction in which graphs are replaced by algebraic curves.

We begin with a brief overview of our main results. Let $\mathcal{I}(\mathcal{D})$ be the set of isomorphism classes of left ideals in $\mathcal{D}$. Since $\mathcal{D}$ is a Noetherian hereditary domain, every ideal of $\mathcal{D}$ is a projective $\mathcal{D}$-module of rank 1, so $\mathcal{I}(\mathcal{D})$ can be equivalently defined as the set of isomorphism classes of such modules. The Grothendieck group $K_0(\mathcal{D})$ of finite rank projective $\mathcal{D}$-modules is isomorphic to the (algebraic) K-group $K_0(X)$ of X, while $K_0(X) \cong \mathbb{Z} \oplus \mathrm{Pic}(X)$, where $\mathrm{Pic}(X)$ is the Picard group of X. Combining these isomorphisms, we may assign to each ideal class $[M] \in \mathcal{I}(\mathcal{D})$ an element of $\mathrm{Pic}(X)$ which determines $[M]$ up to equivalence in $K_0(\mathcal{D})$. In other words, there is a natural map $\gamma : \mathcal{I}(\mathcal{D}) \to \mathrm{Pic}(X)$, whose fibres are

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precisely the *stable* isomorphism classes of ideals in $\mathcal{D}$. Our problem reduces thus to describing the fibres of γ. We approach this problem in two steps. First, we introduce the Calogero–Moser spaces $\mathcal{C}_n(X, \mathcal{I})$ for an arbitrary curve X and a line bundle $\mathcal{I}$ on X, building on the observation of Crawley-Boevey. For any associative algebra B, there is a 'universal' construction of deformed preprojective algebras $\Pi^{\boldsymbol{\lambda}}(B)$ over B, with parameters $\boldsymbol{\lambda} \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(B)$ (see [25] and Section 2.1 below). Using this construction, we define $\mathcal{C}_n(X, \mathcal{I})$ as representation varieties of $\Pi^{\boldsymbol{\lambda}}(B)$ over a triangular matrix extension of the ring $A = \mathcal{O}(X)$ of regular functions on X by the line bundle $\mathcal{I}$. This extension $B = A[\mathcal{I}]$ abstracts the idea of 'framing' a quiver by adjoining a distinguished new vertex '∞' and arrows from ∞; geometrically, it can be thought of as a noncommutative thickening of $\operatorname{Spec}(A \times \mathbb{C}) = X \bigsqcup \mathrm{pt}$. We note that $\mathcal{C}_n(X, \mathcal{I})$ behaves functorially with respect to $\mathcal{I}$; in particular, the quotient spaces $\overline{\mathcal{C}}_n(X, \mathcal{I}) := \mathcal{C}_n(X, \mathcal{I})/\mathrm{Aut}_X(\mathcal{I})$ depend only on the class of $\mathcal{I}$ in $\operatorname{Pic}(X)$. We write $\overline{\mathcal{C}}_n(X)$ for the disjoint union of $\overline{\mathcal{C}}_n(X, \mathcal{I})$ over $\operatorname{Pic}(X)$.

Our first main result is a generalization to an arbitrary X of a known theorem of Wilson (see [51]). The results of the present paper were announced in [10].

Theorem 1.1. *For each $n \geq 0$ and $[\mathcal{I}] \in \operatorname{Pic}(X)$, $\mathcal{C}_n(X, \mathcal{I})$ is a smooth affine irreducible variety of dimension $2n$.*

Now, by functoriality of the $\Pi^{\boldsymbol{\lambda}}$-construction, there is a natural map $\Pi^{\boldsymbol{\lambda}}(B) \to \Pi^1(A)$ lifting the extension $B \to A$. On the other hand, by a theorem of Crawley-Boevey (see [25]), $\Pi^1(A)$ can be identified with the ring $\mathcal{D}$ of differential operators on X. The resulting algebra homomorphism $i : \Pi^{\boldsymbol{\lambda}}(B) \to \mathcal{D}$ relates the module categories of $\Pi^{\boldsymbol{\lambda}}(B)$ and $\mathcal{D}$ in a fairly interesting way. To be precise, we will prove

Theorem 1.2. *The canonical functors $(i^*, i_*, i^!)$ induced by $i : \Pi^{\boldsymbol{\lambda}} \to \mathcal{D}$ on the (bounded) derived categories form a recollement set-up in the sense of [6]:*

$$\begin{tikzcd} \mathcal{D}^b(\mathtt{Mod}\,\mathcal{D}) \arrow[r, "i_*"] & \mathcal{D}^b(\mathtt{Mod}\,\Pi^{\boldsymbol{\lambda}}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{D}^b(\mathtt{Mod}\,U^{\boldsymbol{\lambda}}), \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{1.1}$$

where $U^{\boldsymbol{\lambda}}$ is a (spherical) subalgebra of $\Pi^{\boldsymbol{\lambda}}$ (see Section 4.1).

Originally, the recollement conditions were introduced in [6] in connection with the stratification of a topological space into a closed subspace and its open complement: they axiomatize the induced structure on the derived category $\mathcal{D}(\mathcal{S}h_X)$ of abelian sheaves. In an algebraic setting similar to ours, these conditions were first studied in [24].

The functor i_* yields a fully faithful embedding of $\mathcal{D}^b(\mathtt{Mod}\,\mathcal{D})$ into $\mathcal{D}^b(\mathtt{Mod}\,\Pi^{\boldsymbol{\lambda}})$ as a 'closed stratum', while the induction functor $i^* : \mathcal{D}^b(\mathtt{Mod}\,\Pi^{\boldsymbol{\lambda}}) \to \mathcal{D}^b(\mathtt{Mod}\,\mathcal{D})$ is an algebraic substitute for the restriction of a sheaf to that stratum. This last functor plays a key role in our construction: it transforms irreducible $\Pi^{\boldsymbol{\lambda}}(B)$-modules (viewed as 0-complexes in $\mathcal{D}^b(\mathtt{Mod}\,\Pi^{\boldsymbol{\lambda}})$) to projective $\mathcal{D}$-modules (located in homological degree -1), inducing natural maps

$$\omega_n : \overline{\mathcal{C}}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}].$$

The main result of this paper can now be encapsulated in the following theorem.

Theorem 1.3. *Let X be a smooth affine irreducible curve over $\mathbb{C}$.*

(a) *For each $[\mathcal{I}] \in \operatorname{Pic}(X)$, amalgamating the maps ω_n for all $n \geq 0$ yields a bijective correspondence*

$$\omega : \bigsqcup_{n \geq 0} \overline{\mathcal{C}}_n(X, \mathcal{I}) \xrightarrow{\sim} \gamma^{-1}[\mathcal{I}].$$

(b) *There is a natural action on $\overline{\mathcal{C}}_n(X)$ of the Picard group $\operatorname{Pic}(\mathcal{D})$ of the category of $\mathcal{D}$-modules, and the maps $\omega_n : \overline{\mathcal{C}}_n(X) \to \mathcal{I}(\mathcal{D})$ are equivariant under this action for all $n \geq 0$.*

Part (a) of Theorem 1.3 gives a geometric description of the fibration γ over a given $[\mathcal{I}] \in \operatorname{Pic}(X)$. In the special case when X is the affine line, $\operatorname{Pic}(X)$ is trivial: there is only one fibre, and it is shown in [13] that ω agrees with the Calogero–Moser map constructed in [15,16]. Part (b) generalizes another aspect of the Calogero–Moser correspondence for the Weyl algebra: the equivariance of the Calogero–Moser map under the action of the automorphism group $\mathrm{Aut}_{\mathbb{C}}(A_1)$ (which is known to be isomorphic to $\operatorname{Pic}(A_1)$, see [49]). The importance of this result is that it allows one to classify the algebras Morita equivalent to $\mathcal{D}$ up to isomorphism. Precisely, Theorem 1.3(b) implies that the isomorphism classes of domains $\mathcal{D}'$ Morita equivalent to $\mathcal{D}$ are in one-to-one correspondence with the orbits of $\operatorname{Pic}(\mathcal{D})$ on the Calogero–Moser spaces $\overline{\mathcal{C}}_n(X)$. For example, for $n = 0$, we have $\overline{\mathcal{C}}_0(X) = \operatorname{Pic}(X)$, and the action of $\operatorname{Pic}(\mathcal{D})$ is transitive on $\operatorname{Pic}(X)$ (see Proposition 4.1 below); this implies a theorem of Cannings and Holland ([20], Theorem 1.10) that $\mathcal{D}' \cong \mathcal{D}$ if and only if $\mathcal{D}' \cong \mathrm{End}_{\mathcal{D}}(\mathcal{I}\mathcal{D})$ for some line bundle $\mathcal{I}$. For an arbitrary $n > 0$, the structure of orbits of $\operatorname{Pic}(\mathcal{D})$ in $\overline{\mathcal{C}}_n(X)$ is complicated; however, one can still define a complete set of isomorphism invariants for the algebras $\mathcal{D}'$ in terms of the Hochschild homology of $\Pi^{\boldsymbol{\lambda}}(B)$. We will discuss this construction elsewhere.

We will now explain how our results relate to earlier work. The problem of classifying ideals of $\mathcal{D}(X)$ for a smooth affine curve X was first addressed by Cannings and Holland (see [19,20]) who identified the space $\mathcal{I}(\mathcal{D})$ with a certain infinite-dimensional Grassmannian. In the special case when $X = \mathbb{A}^1$, this Grassmannian was introduced independently (and for a different purpose) by Wilson [52], who called it the adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}$. Motivated by earlier work on integrable systems [1,22,40,36], Wilson showed (see [51]) that $\mathrm{Gr}^{\mathrm{ad}}$ can be decomposed into a countable union of smooth varieties $\mathcal{C}_n$ which are now called the Calogero–Moser spaces. It is important to understand that the Calogero–Moser decomposition is entirely different from the obvious stratification of $\mathrm{Gr}^{\mathrm{ad}}$ by finite-dimensional Grassmannians considered in [20]. Its relevance for the Weyl algebra $A_1(\mathbb{C})$ became clear in [15], where it was shown that, under the Cannings–Holland bijection, the spaces $\mathcal{C}_n$ correspond to the orbits of the natural action of the Dixmier group $\mathrm{Aut}_{\mathbb{C}}(A_1)$ on $\mathcal{I}(A_1)$. A different approach to the problem of classifying ideals of A_1, which does not use $\mathrm{Gr}^{\mathrm{ad}}$, was developed in [16]. The main idea of [16] – to use noncommutative projective geometry (specifically, a noncommutative version of Beilinson's equivalence) – was inspired by [42] and [35] and was later generalized to many other classes of quantum algebras (see [3,4], [45,46,9] and references therein). While the present paper was in preparation, a new interesting paper [8] by Ben-Zvi and Nevins has appeared. In [8], the authors use a noncommutative Beilinson equivalence to classify torsion-free $\mathcal{D}$-modules on projective curves. Although this last problem is similar to the one addressed in the present paper, our methods and results are different. Apart from describing explicitly the space $\mathcal{I}(\mathcal{D})$ of ideals, we also describe the action of the Picard group on $\mathcal{I}(\mathcal{D})$ and prove the equivariance of the Calogero–Moser correspondence. Comparing our constructions to those of [8] is an interesting problem which will hopefully be clarified elsewhere. We also mention that the methods of the present paper apply to a more general class of formally smooth algebras, including the path algebras of quivers. Some of these versions of the Calogero–Moser correspondence will be a subject of a forthcoming work. Finally, in the existing literature, there are (at least) two other definitions of Calogero–Moser spaces associated to curves. The first one, due to V. Ginzburg, employs the classical Hamiltonian reduction (see [33], or [8], Def. 1.2) and is, in fact, closely related to ours (see Remark in the end of Section 3.3). The second, due to Etingof (see [31], Example 2.19), is given in terms of generalized Cherednik algebras (in the style of [32]). We will discuss the relation of Etingof's definition to ours in [12].

We now proceed with a summary of the contents of the paper. Section 2 is preliminary: it introduces notation and reviews the material needed for the rest of the paper. While most results in this section are known, some are (apparently) new. In particular, Theorem 2.2, Propositions 2.1 and 2.2 did not appear in the literature in this form and generality. In Section 3, after recollections on differential operators (Section 3.1) and K-theoretic classification of ideals of $\mathcal{D}$ (Section 3.2), we define the Calogero–Moser spaces $\mathcal{C}_n(X, \mathcal{L})$ and establish their basic properties, including Theorem 1.1. The main results of the paper are gathered in Section 4. First, in Section 4.1, we explain the relation between the algebras $\Pi^{\lambda}(B)$ and $\mathcal{D}$, including Theorem 1.2. Then, in Sections 4.2 and 4.3, we describe the action of the Picard group $\mathrm{Pic}(\mathcal{D})$ on the Calogero–Moser spaces $\overline{\mathcal{C}}_n(X)$ and state our main Theorem 4.2, which is a more precise version of Theorem 1.3. The proof of Theorem 4.2 occupies the whole of Section 5. We refer the reader to the introduction of that section for a summary of the proof. In Section 6, we give an alternative description of the map ω and consider a number of explicit examples. Perhaps, the most interesting example is that of a general plane curve (see Section 6.2.3). In this case, the varieties $\mathcal{C}_n(X, \mathcal{L})$ can be described in terms of matrices generalizing the classical Calogero–Moser matrices, and the map ω is given by an explicit formula involving characteristic polynomials of these matrices (see (6.11)). This last formula can be viewed as a generalization of Wilson's formula for the rational Baker function of the KP integrable hierarchy (see [51]); however, the way we derive it is different from that of [51]. Our method extends the earlier calculations of [11] in the case of $A_1(\mathbb{C})$.

The last section of the paper is an appendix written by G. Wilson. It clarifies the relation between deformed preprojective algebras and rings of differential operators on curves. As explained above, our map ω is naturally induced by the algebra extension $i : \Pi^{\lambda}(B) \to \mathcal{D}$. Unfortunately, this extension is not entirely canonical: it depends on the choice of an identification of $\Pi^1(A)$ with $\mathcal{D} = \mathcal{D}(X)$. By a theorem of Crawley-Boevey (see [25], Theorem 4.7), $\Pi^1(A)$ is indeed isomorphic to $\mathcal{D}$ as a filtered algebra, but, in general, there seems to be no natural isomorphism between these algebras. To remedy this problem, one should replace $\mathcal{D}$ by the ring $\mathcal{D}(\Omega_X^{1/2})$ of twisted differential operators on half-forms on X. As was first observed by Ginzburg (see [34], Section 13.4), $\Pi^1(A)$ is *canonically* isomorphic to $\mathcal{D}(\Omega_X^{1/2})$; however, the construction of the isomorphism depends on a fact (Proposition A.1 below) which is not proved in [34]. A simple proof is given in the Appendix which may be read independently of the rest of the paper. To avoid confusion we note that this proof naturally works in the complex analytic setting: when X is a compact Riemann surface (smooth projective curve); however, all constructions, including the key isomorphism of Proposition A.1, are sheaf-theoretic, so the results of the Appendix apply to the affine curves as well.

Notation and conventions

Throughout this paper, we work over the base field $\mathbb{C}$. Unless otherwise specified, an algebra means an associative algebra over $\mathbb{C}$, a module over an algebra A means a *left* module over A, and $\mathtt{Mod}(A)$ denotes the category of such modules. All bimodules over algebras are assumed to be symmetric over $\mathbb{C}$, and we use the abbreviation $\otimes$ for $\otimes_{\mathbb{C}}$ whenever it is convenient.

2. Preliminaries

2.1. Deformed preprojective algebras

If A is an algebra, its tensor square $A^{\otimes 2}$ has two commuting bimodule structures: one is defined by $a.(x\otimes y).b = ax\otimes yb$ and the other by $a.(x\otimes y).b = xb\otimes ay$, where $a, b \in A$. We will refer to these structures as *outer* and *inner*, respectively. Any bimodule over A can be viewed as either left or right module over the enveloping algebra $A^{\mathrm{e}} := A \otimes A^{\mathrm{o}}$; if we interpret the outer bimodule structure on $A^{\otimes 2}$ as a left A^{e}-module structure and the inner as a right one, then the canonical map $A^{\otimes 2} \to A^{\mathrm{e}}$ is an isomorphism of A^{e}-bimodules. We will often use this isomorphism to identify $A^{\otimes 2} \cong A^{\mathrm{e}}$.

Following [28], we let $\mathbb{D}\mathrm{er}(A) := \mathrm{Der}(A, A^{\otimes 2})$ denote the space of linear derivations $A \to A^{\otimes 2}$ taken with respect to the outer bimodule structure on $A^{\otimes 2}$. This space is a bimodule with respect to the inner structure, so we can form the tensor algebra $T_A\mathbb{D}\mathrm{er}(A)$. Now, in $\mathbb{D}\mathrm{er}(A)$, there is a canonical derivation $\Delta = \Delta_A$, sending $x \in A$ to $(x \otimes 1 - 1 \otimes x) \in A^{\otimes 2}$. For any $\lambda \in A$, we can consider then the 2-sided ideal $\langle \Delta - \lambda \rangle$ in $T_A\mathbb{D}\mathrm{er}(A)$ and define $\Pi^{\lambda}(A) := T_A\mathbb{D}\mathrm{er}(A)/\langle \Delta - \lambda \rangle$. It turns out that, up to isomorphism, the algebra $\Pi^{\lambda}(A)$ depends only on the class of λ in the Hochschild homology $H_0(A) := A/[A, A]$ (see [25], Lemma 1.2). Moreover, instead of elements of $H_0(A)$, it is convenient to parametrize $\Pi^{\lambda}(A)$ by the elements of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$, relating this last vector space to $H_0(A)$ via a Chern character map. To be precise, let $\mathrm{Tr}_A : K_0(A) \to H_0(A)$ be the map, sending the class of a projective module P to the class of the trace of any idempotent $e \in \mathtt{Mat}(n, A)$, satisfying $P \cong A^n e$. By additivity, this extends to a linear map $\mathbb{C} \otimes_{\mathbb{Z}} K_0(A) \to H_0(A)$ to be denoted also Tr_A. Following [25], we call the elements of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$ *weights* and define the *deformed preprojective algebra of weight* $\boldsymbol{\lambda} \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$ by

$$\Pi^{\boldsymbol{\lambda}}(A) := T_A\mathbb{D}\mathrm{er}(A)/\langle \Delta - \lambda \rangle, \tag{2.1}$$

where $\lambda \in A$ is any lifting of $\mathrm{Tr}_A(\boldsymbol{\lambda})$ to A. Note, if A is commutative, then $H_0(A) = A$, and λ is uniquely determined by $\mathrm{Tr}_A(\boldsymbol{\lambda})$.

The algebras $\Pi^{\boldsymbol{\lambda}}(A)$ are usually ill-behaved unless one imposes some 'smoothness' conditions on A. In this paper, following [38], we call an algebra A *smooth* if it is quasi-free and finitely generated. By definition (see [30]), a quasi-free algebra behaves like a free algebra with respect to nilpotent extensions (in the sense that any ring homomorphism $A \to R/I$, where I is a nilpotent ideal in R, can be lifted to a homomorphism $A \to R$). Over the complex numbers, quasi-free algebras can be characterized cohomologically as the algebras having dimension ≤ 1 with respect to Hochschild cohomology. This implies that if A is a smooth algebra, the kernel of its multiplication map (denoted by Ω^1A) is a f. g. projective bimodule (see [30], Proposition 3.3).

For basic properties and examples of the algebras $\Pi^{\boldsymbol{\lambda}}(A)$ the reader is referred to [25]. Here, we state only one important theorem from [25] which will play a role in our construction. We recall that a ring homomorphism $i : B \to A$ is called *pseudo-flat* if $\mathrm{Tor}_1^B(A, A) = 0$. We also recall that any ring homomorphism $i : B \to A$ induces a homomorphism of abelian groups $i_* : K_0(B) \to K_0(A)$ which extends (by linearity) to a map of $\mathbb{C}$-vector spaces $i_* : \mathbb{C} \otimes_{\mathbb{Z}} K_0(B) \to \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$.

Theorem 2.1 (*[25], Theorem 9.3 and Corollary 9.4*). *Let $i : B \to A$ be a pseudo-flat ring epimorphism. Then, for any $\boldsymbol{\lambda} \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(B)$, there is a canonical algebra map $i : \Pi^{\boldsymbol{\lambda}}(B) \to \Pi^{\boldsymbol{\lambda}'}(A)$, where $\boldsymbol{\lambda}' = i_*(\boldsymbol{\lambda})$. If B is smooth, then i is also a pseudo-flat epimorphism, and the diagram*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ \Pi^{\boldsymbol{\lambda}}(B) \arrow[r] & \Pi^{\boldsymbol{\lambda}'}(A) \end{tikzcd} \tag{2.2}$$

is a push-out in the category of rings.

We now prove a few general results on representations of deformed preprojective algebras which may be of independent interest. Our first lemma is probably well known to the experts (see, for example, [28]); we record it to fix the notation.

Lemma 2.1. *If A is smooth, then Δ lies in the commutator space $[A, \mathbb{D}\mathrm{er}(A)]$ of the bimodule $\mathbb{D}\mathrm{er}(A)$.*

Proof. Composing the multiplication map $\mu : A^{\otimes 2} \to A$ with derivations $A \to A^{\otimes 2}$ yields a linear map $\mu_* : \mathbb{D}\mathrm{er}(A) \to \mathrm{Der}(A)$, with $\Delta \in \mathrm{Ker}(\mu_*)$. This map factors through the natural projection $\mathbb{D}\mathrm{er}(A) \twoheadrightarrow \mathbb{D}\mathrm{er}(A)_\natural$, where $\mathbb{D}\mathrm{er}(A)_\natural := \mathbb{D}\mathrm{er}(A)/[A, \mathbb{D}\mathrm{er}(A)]$. If A is smooth, the induced map $\bar{\mu}_* : \mathbb{D}\mathrm{er}(A)_\natural \to \mathrm{Der}(A)$ is an isomorphism. Indeed, identifying $A^{\otimes 2} \cong A^{\mathrm{e}}$ and writing $\Omega^1A \subseteq A^{\mathrm{e}}$ for $\mathrm{Ker}(\mu)$, we have $\mathrm{Der}(A) \cong \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1A, A)$ and $\mathbb{D}\mathrm{er}(A) \cong \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1A, A^{\mathrm{e}})$. Under the last isomorphism, the bimodule structure on $\mathbb{D}\mathrm{er}(A)$ corresponds to the natural right A^{e}-module structure on $(\Omega^1A)^\star := \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1A, A^{\mathrm{e}})$ and $\mathbb{D}\mathrm{er}(A)_\natural \cong (\Omega^1A)^\star \otimes_{A^{\mathrm{e}}} A$. The quotient map $\bar{\mu}_*$ now becomes $(\Omega^1A)^\star \otimes_{A^{\mathrm{e}}} A \to \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1A, A)$. Since A is smooth, Ω^1A is a f. g. projective A^{e}-module, so the last map is an isomorphism. This implies that $\mathrm{Ker}(\mu_*) = [A, \mathbb{D}\mathrm{er}(A)]$, and hence $\Delta \in [A, \mathbb{D}\mathrm{er}(A)]$. $\square$

Example 2.1. Let $A = \mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle$ be a free algebra on $n \geq 1$ generators. Then $\mathbb{D}\mathrm{er}(A)$ is a free bimodule of rank n; for a basis in $\mathbb{D}\mathrm{er}(A)$, it is natural to take the derivations $y_i : A \to A \otimes A$, $i = 1, 2, \ldots, n$, defined by

$$y_i(x_j) := \delta_{ij}\,(1 \otimes 1).$$

One can easily check that the canonical derivation on A is expressed in terms of this basis by

$$\Delta = \sum_{i=1}^{n} [y_i, x_i].$$

The free algebra A can be viewed as the path algebra of the quiver consisting of one vertex and n loops. A similar explicit presentation of the bimodule $\mathbb{D}\mathrm{er}(A)$ can be given for the path algebra of an arbitrary quiver (see [25], Theorem 3.1).

In this paper, we will describe $\mathbb{D}\mathrm{er}(A)$ in the case when A is the coordinate ring of a nonsingular affine curve and its triangular matrix extension (see Lemmas 5.2 and 6.2, respectively).

For any $\lambda \in A$, the algebra $\Pi^\lambda(A)$ is an A-ring: it is equipped with a canonical algebra homomorphism $A \to \Pi^\lambda(A)$. Every representation of $\Pi^\lambda(A)$ can thus be regarded as a representation of A. Conversely, given a representation of A, one can ask whether it lifts to a representation of $\Pi^\lambda(A)$. The following theorem provides a simple homological criterion for the existence and uniqueness of such liftings.

Theorem 2.2. *Let A be a smooth algebra, and let $\varrho : A \to \mathrm{End}(V)$ be a representation of A on a (not necessarily finite-dimensional) vector space V. Then ϱ can be extended to a representation of $\Pi^\lambda(A)$ if and only if the homology class of $\varrho(\lambda)$ in $H_0(A, \mathrm{End}\, V)$ is zero, i.e. $\varrho(\lambda) \in [\varrho(A), \mathrm{End}(V)]$. If it exists, an extension of ϱ to $\Pi^\lambda(A)$ is unique if and only if $H_1(A, \mathrm{End}\, V) = 0$.*

Proof. We will use the notation of Lemma 2.1. Thus, for a fixed $\lambda \in A$, we identify $\Pi^\lambda(A) = T_A(\Omega^1 A)^\star/\langle \Delta_A - \lambda\rangle$, with $\Delta_A \in (\Omega^1 A)^\star$ corresponding to the natural inclusion $\Omega^1 A \hookrightarrow A^{\mathrm{e}}$. A representation $\varrho : A \to \mathrm{End}(V)$ can be extended then to a representation of $\Pi^\lambda(A)$ if and only if there is an A-ring map $\tilde{\varrho} : T_A(\Omega^1 A)^\star \to \mathrm{End}(V)$, such that $\tilde{\varrho}(\Delta_A) = \varrho(\lambda)$. By the universal property of tensor algebras, such a map is uniquely determined by its restriction to $(\Omega^1 A)^\star$. Thus, regarding $\mathrm{End}(V)$ as a bimodule over A via ϱ, we conclude that ϱ lifts to $\Pi^\lambda(A)$ iff there is $\tilde{\varrho} \in \mathrm{Hom}_{A^{\mathrm{e}}}((\Omega^1 A)^\star, \mathrm{End}\, V)$, mapping Δ_A to $\varrho(\lambda)$. Here, the bimodule $\mathrm{End}(V)$ is interpreted as a *right* A^{e}-module.

Now, since A is smooth, the canonical map $\Omega^1 A \to (\Omega^1 A)^{\star\star}$ is an isomorphism, and we can identify $\mathrm{Hom}_{A^{\mathrm{e}}}((\Omega^1 A)^\star, \mathrm{End}\, V) \cong \mathrm{End}(V) \otimes_{A^{\mathrm{e}}} \Omega^1 A$. Under this identification, the condition $\tilde{\varrho}(\Delta_A) = \varrho(\lambda)$ becomes

$$\exists f_i \otimes d_i \in \mathrm{End}(V) \otimes_{A^{\mathrm{e}}} \Omega^1 A : \sum_i f_i\, \Delta_A(d_i) = \varrho(\lambda). \tag{2.3}$$

Tensoring the exact sequence of A^{e}-modules $0 \to \Omega^1 A \to A^{\mathrm{e}} \to A \to 0$ with $\mathrm{End}(V)$, we get

$$0 \to H_1(A, \mathrm{End}\, V) \to \mathrm{End}(V) \otimes_{A^{\mathrm{e}}} \Omega^1 A \xrightarrow{\partial} \mathrm{End}(V) \xrightarrow{p} H_0(A, \mathrm{End}\, V) \to 0\,, \tag{2.4}$$

with map in the middle given by $\partial : f \otimes d \mapsto f\, \Delta_A(d)$, and p being the canonical projection. The condition (2.3) now says that $\varrho(\lambda) \in \mathrm{Im}(\partial)$, and, by exactness of (2.4), this is equivalent to $\varrho(\lambda) \in \mathrm{Ker}(p)$. Thus, ϱ can be extended to $\Pi^\lambda(A)$ if and only $\varrho(\lambda)$ vanishes in $H_0(A, \mathrm{End}\, V)$.

The fibre of ∂ over $\varrho(\lambda) \in \mathrm{End}(V)$ consists of different liftings of the given action ϱ to $\Pi^\lambda(A)$. Again, by exactness of (2.4), this fibre can be identified with $H_1(A, \mathrm{End}\, V)$. In particular, if ϱ admits an extension to $\Pi^\lambda(A)$, this extension is unique if and only if $H_1(A, \mathrm{End}\, V) = 0$. $\square$

As an immediate corollary of Theorem 2.2, we get

Corollary 2.1. *If $\boldsymbol{\lambda} \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$, then $\varrho : A \to \mathrm{End}(V)$ can be extended to $\Pi^{\boldsymbol{\lambda}}(A)$ if and only if $\varrho_* \mathrm{Tr}_A(\boldsymbol{\lambda}) = 0$, where $\varrho_* : H_0(A) \to H_0(A, \mathrm{End}\, V)$ is the map induced by ϱ on Hochschild homology.*

We now apply Theorem 2.2 to finite-dimensional representations. The next result is a generalization of [27], Theorem 3.3, which deals with path algebras of quivers.

Proposition 2.1. *Let A be a smooth algebra, and let $\varrho : A \to \mathrm{End}(V)$ be a representation of A on a finite-dimensional vector space V. Then ϱ lifts to a representation of $\Pi^\lambda(A)$ if and only if the trace of $\varrho(\lambda)$ on any A-module direct summand of V is zero. Moreover, if $\varrho \in \mathrm{Rep}(A, V)$ lifts, then the fibre $\pi^{-1}(\varrho)$ of the canonical map $\pi : \mathrm{Rep}(\Pi^\lambda(A), V) \to \mathrm{Rep}(A, V)$ is isomorphic to $\mathrm{Ext}^1_A(V, V)^*$.*

Proof. The trace pairing on $\mathrm{End}(V)$ yields a linear isomorphism $\mathrm{End}(V) \xrightarrow{\sim} \mathrm{End}(V)^*$ which is a bimodule map with respect to the natural bimodule structures on $\mathrm{End}(V)$ and $\mathrm{End}(V)^*$. This isomorphism restricts to $\mathrm{End}_A(V) \xrightarrow{\sim} H_0(A, \mathrm{End}\, V)^*$ which, upon dualizing with $\mathbb{C}$, becomes

$$H_0(A, \mathrm{End}\, V) \xrightarrow{\sim} \mathrm{End}_A(V)^*, \qquad \bar{f} \mapsto [e \mapsto \mathrm{tr}_V(ef)]. \tag{2.5}$$

Now, let $\varrho : A \to \mathrm{End}(V)$ be a representation of A on V that lifts to $\Pi^\lambda(A)$, and suppose that V has a direct A-linear summand, say W. By Theorem 2.2, the class of $\varrho(\lambda)$ in $H_0(A, \mathrm{End}\, V)$ is zero, and hence so is its image under (2.5). Taking $e \in \mathrm{End}_A(V)$ to be a projection onto W, we get $\mathrm{tr}_V[e\varrho(\lambda)] = \mathrm{tr}_W[\varrho(\lambda)] = 0$ which proves the first implication of the theorem.

For the converse, it suffices to consider only the indecomposable representations $\varrho : A \to \mathrm{End}(V)$. By Fitting's Lemma, $\mathrm{End}_A(V)$ is then a local ring, which means that every $e \in \mathrm{End}_A(V)$ can be written in the form $e = c\, \mathrm{Id}_V + \theta$, where $c \in \mathbb{C}$ and θ is nilpotent. Now, if we assume that $\mathrm{tr}_V[\varrho(\lambda)] = 0$, then

$$\mathrm{tr}_V[e\varrho(\lambda)] = c\, \mathrm{tr}_V[\varrho(\lambda)] + \mathrm{tr}_V[\theta\varrho(\lambda)] = \mathrm{tr}_V[\theta\varrho(\lambda)]. \tag{2.6}$$

Since $\mathrm{End}_A(V)$ is the commutant of the image of ϱ in $\mathrm{End}(V)$, $[\theta, \varrho(\lambda)] = 0$. So θ being nilpotent implies that $\theta\varrho(\lambda)$ is nilpotent and therefore traceless. It follows from (2.6) that $\mathrm{tr}_V[e\varrho(\lambda)] = 0$ for all $e \in \mathrm{End}_A(V)$. The class of $\varrho(\lambda)$ in

$H_0(A,\ \mathrm{End}\,V)$ lies thus in the kernel of (2.5) and hence is zero. By Theorem 2.2, we conclude that ϱ lifts to a representation of $\Pi^\lambda(A)$.

For the last statement, note that $\pi^{-1}(\varrho) \cong H_1(A,\ \mathrm{End}\,V)$ by exactness of (2.4). On the other hand,

$$H_1(A,\ \mathrm{End}\,V) \cong \mathrm{Tor}_1^A(V^*,\ V) \cong \mathrm{Ext}_A^1(V, V)^*, \tag{2.7}$$

which is standard homological algebra (see [21], Corollary 4, p. 170, and Proposition VI, 5.3). □

Remark. In the special case when A is the path algebra of a quiver, Proposition 2.1 was proven earlier, by a different method, in [27]. With identifications (2.5) and (2.7), our basic sequence (2.4) becomes

$$0 \to \mathrm{Ext}_A^1(V,\ V)^* \to \mathrm{End}(V) \otimes_{A^e} \Omega^1 A \to \mathrm{End}(V) \to \mathrm{End}_A(V)^* \to 0, \tag{2.8}$$

which, in the quiver case, agrees with [27], Lemma 3.1.

2.2. One-point extensions

If A is a unital associative algebra, and I a left module over A, we define the *one-point extension* of A by I to be the ring of triangular matrices

$$A[I] := \begin{pmatrix} A & I \\ 0 & \mathbb{C} \end{pmatrix} \tag{2.9}$$

with matrix addition and multiplication induced from the module structure of I. Clearly, $A[I]$ is a unital associative algebra, with identity element being the identity matrix. There are two distinguished idempotents in $A[I]$: namely

$$e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.10}$$

If A is indecomposable (e.g., A is a commutative integral domain), then (2.10) form a complete set of primitive orthogonal idempotents in $A[I]$.

A module over $A[I]$ can be identified with a triple $\boldsymbol{V} = (V,\ V_\infty,\ \varphi)$, where V is an A-module, V_∞ is a $\mathbb{C}$-vector space and $\varphi : I \otimes V_\infty \to V$ is an A-module map. Using the standard matrix notation, we will write the elements of $\boldsymbol{V}$ as column vectors $(v, w)^T$ with $v \in V$ and $w \in V_\infty$; the action of $A[I]$ is then given by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} a.v + \varphi(b \otimes w) \\ cw \end{pmatrix}.$$

Now, if $\boldsymbol{U} = (U,\ U_\infty,\ \varphi_U)$ and $\boldsymbol{V} = (V,\ V_\infty,\ \varphi_V)$ are two $A[I]$-modules, a homomorphism $\boldsymbol{U} \to \boldsymbol{V}$ is determined by a pair of maps (f, f_∞), with $f \in \mathrm{Hom}_A(U, V)$ and $f_\infty \in \mathrm{Hom}_{\mathbb{C}}(U_\infty, V_\infty)$, making the following diagram commutative

$$\begin{tikzcd} I \otimes U_\infty \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{Id} \otimes f_\infty"'] & U \arrow[d, "f"] \\ I \otimes V_\infty \arrow[r, "\varphi_V"] & V \end{tikzcd} \tag{2.11}$$

If $\boldsymbol{V}$ is finite-dimensional, with $\dim_{\mathbb{C}} V = n$ and $\dim_{\mathbb{C}} V_\infty = n_\infty$, we call $\boldsymbol{n} = (n,\ n_\infty)$ the *dimension vector* of $\boldsymbol{V}$.

The next lemma gathers together basic properties of one-point extensions.

Lemma 2.2. (1) *$A[I]$ is canonically isomorphic to $T_{\tilde{A}}(I)$, where $\tilde{A} := A \times \mathbb{C}$.*

(2) *If A is smooth and I is a f. g. projective A-module, then $A[I]$ is smooth.*

(3) *$I \mapsto A[I]$ is a functor from $\mathtt{Mod}(A)$ to the category of associative algebras.*

(4) *The natural projection $i : A[I] \to A$ is a flat ring epimorphism.*

(5) *There is an isomorphism of abelian groups $K_0(A[I]) \cong K_0(A) \oplus \mathbb{Z}$.*

Proof. (1) We identify $\tilde{A}$ with the subalgebra of diagonal matrices in $A[I]$ and I with the complementary nilpotent ideal $\tilde{I} \subset A[I]$:

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & \mathbb{C} \end{pmatrix}, \qquad \tilde{I} := \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}. \tag{2.12}$$

By the universal property of tensor algebras, the inclusions $\tilde{A} \hookrightarrow A[I]$ and $\tilde{I} \hookrightarrow A[I]$ can then be extended to an algebra map $\phi : T_{\tilde{A}}(\tilde{I}) \to A[I]$ which is a required isomorphism.

(2) By (1) and [30], Proposition 5.3, it suffices to show that $\tilde{I}$ is a projective $\tilde{A}$-bimodule. But if I is a projective A-module, then it is isomorphic to a direct summand of a free module $A \otimes V$ and $\tilde{I}$ is isomorphic to a direct summand of $\tilde{A}e \otimes V \otimes e_\infty \tilde{A}$. The latter is a projective $\tilde{A}$-bimodule, since it is a direct summand of $\tilde{A} \otimes V \otimes \tilde{A}$.

(3) Any A-module map $f : I_1 \to I_2$ gives rise to an $\tilde{A}$-bimodule map $\tilde{f} : \tilde{I}_1 \to \tilde{I}_2$. Identifying $A[I_1] = T_{\tilde{A}}(\tilde{I}_1)$ and $A[I_2] = T_{\tilde{A}}(\tilde{I}_2)$, we may extend $I \mapsto A[I]$ to morphisms by $A[f] := T_{\tilde{A}}(\tilde{f})$. As $T_{\tilde{A}}$ is a functor on bimodules, the result follows.

(4) The map i is given by

$$i : A[I] \to A, \qquad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a. \tag{2.13}$$

It is immediate from (2.13) that $A \cong A[I]\, e$ as a left $A[I]$-module via i. Since e is an idempotent, $A[I]\, e$ is projective and hence flat.

(5) The diagonal projection $\tilde{i} : A[I] \to \tilde{A}$ has a nilpotent kernel (equal to $\tilde{I}$). By [5], Proposition IX.1.3, it then induces isomorphisms $K_i(A[I]) \cong K_i(\tilde{A})$ for all i. In particular, $K_0(A[I]) \cong K_0(A) \oplus \mathbb{Z}$. □

We will also need the next lemma relating homological properties of A and $A[I]$.

Lemma 2.3. *Let A be a finitely generated hereditary algebra, and let $B := A[I]$ be the one-point extension of A by a f. g. projective A-module. Then, for any finite-dimensional B-modules $\boldsymbol{U} = (U, U_\infty)$ and $\boldsymbol{V} = (V, V_\infty)$, we have*

$$\chi_B(\boldsymbol{U}, \boldsymbol{V}) = \chi_A(U, V) + \dim(U_\infty)\,[\dim(V_\infty) - \dim \operatorname{Hom}_A(I, V)], \tag{2.14}$$

where χ_A and χ_B denote the Euler characteristics for the Ext-groups over the algebras A and B respectively.

Proof. By [7], Théorème 1.1 (bis), there is a 5-term exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_B(\boldsymbol{U}, \boldsymbol{V}) \to \operatorname{Hom}_A(U, V) \oplus \operatorname{Hom}_{\mathbb{C}}(U_\infty, V_\infty) \to \\ &\to \operatorname{Hom}_{\mathbb{C}}(U_\infty,\ \operatorname{Hom}_A(I, V)) \to \operatorname{Ext}^1_B(\boldsymbol{U}, \boldsymbol{V}) \to \operatorname{Ext}^1_A(U, V) \to 0, \end{aligned} \tag{2.15}$$

and isomorphisms $\operatorname{Ext}^k_B(\boldsymbol{U}, \boldsymbol{V}) = \operatorname{Ext}^k_A(U,\ V) = 0$ for all $k \geq 2$, since A is hereditary. Now, since A is finitely generated, $\operatorname{Hom}_A(U, V)$ and $\operatorname{Ext}^1_A(U, V)$ are finite-dimensional whenever U and V are finite-dimensional. It follows from (2.15) that $\chi_B(\boldsymbol{U}, \boldsymbol{V})$ is well defined and related to $\chi_A(U, V)$ by (2.14). □

2.3. Representation varieties

We recall the definition of representation varieties in the form they appear in representation theory of associative algebras (see [39], Chapter II, Section 2.7).

Let R be a finitely generated algebra. Fix S, a finite-dimensional semisimple subalgebra of R, and V, a finite-dimensional S-module. By definition, the *representation variety* $\operatorname{Rep}_S(R, V)$ of R over S parametrizes all R-module structures on the vector space V extending the given S-module structure on it. The S-module structure on V determines an algebra homomorphism $S \to \operatorname{End}(V)$ making $\operatorname{End}(V)$ an S-algebra. A point of $\operatorname{Rep}_S(R, V)$ can thus be interpreted as an S-algebra map $\varrho : R \to \operatorname{End}(V)$, and the tangent vectors at ϱ can be identified with S-linear derivations $R \to \operatorname{End}(V)$, i. e.

$$T_\varrho \operatorname{Rep}_S(R, V) \cong \operatorname{Der}_S(R,\ \operatorname{End} V). \tag{2.16}$$

If $S = \mathbb{C}$, we simply write $\operatorname{Rep}(R, V)$ for $\operatorname{Rep}_{\mathbb{C}}(R, V)$. Clearly, $\operatorname{Rep}(R, V)$ is an affine variety.[1] For any semisimple $S \subseteq R$, $\operatorname{Rep}_S(R, V)$ can then be identified with a fibre of the canonical morphism $\operatorname{Rep}(R, V) \to \operatorname{Rep}(S, V)$, and hence it is an affine variety as well.

The group $\operatorname{Aut}_S(V)$ of S-linear automorphisms of V acts on $\operatorname{Rep}_S(R, V)$ in the natural way, with scalars $\mathbb{C}^* \subseteq \operatorname{Aut}_S(V)$ acting trivially. We set $\mathrm{G}_S(V) := \operatorname{Aut}_S(V)/\mathbb{C}^*$. The orbits of $\mathrm{G}_S(V)$ on $\operatorname{Rep}_S(R, V)$ are in 1-1 correspondence with isomorphism classes of R-modules which are isomorphic to V as S-modules. The stabilizer of a point $\varrho : R \to \operatorname{End}(V)$ in $\operatorname{Rep}_S(R, V)$ is canonically isomorphic to $\operatorname{Aut}_R(V_\varrho)/\mathbb{C}^* \subseteq \mathrm{G}_S(V)$, where V_ϱ is the left R-module corresponding to ϱ. The space $\operatorname{Rep}_S(R, V)/\!/\, \mathrm{G}_S(V)$ of closed orbits in $\operatorname{Rep}_S(R, V)$ is an affine variety, whose points are in bijection with isomorphism classes of semisimple R-modules M isomorphic to V as S-modules (*cf.* [39], II.2.7, Theorem 3; or [34], Theorem 12.2.3).

Typically, representation varieties of R are defined over subalgebras $S = \oplus_{i \in I} \mathbb{C}\, e_i \subset R$ spanned by idempotents. A finite-dimensional S-module is then isomorphic to a direct sum $\mathbb{C}^{\boldsymbol{n}} := \oplus_{i \in I} \mathbb{C}^{n_i}$, each e_i acting as the projection onto the i-th component. The variety $\operatorname{Rep}_S(R, \mathbb{C}^{\boldsymbol{n}})$, which we simply denote by $\operatorname{Rep}_S(R, \boldsymbol{n})$ in this case, parametrizes all algebra maps $R \to \operatorname{End}(\mathbb{C}^{\boldsymbol{n}})$ sending e_i to the projection onto $\mathbb{C}^{n_i}$. The group $\mathrm{G}_S(\mathbb{C}^{\boldsymbol{n}})$ (to be denoted $\mathrm{G}_S(\boldsymbol{n})$) is isomorphic to $\prod_{i \in I} \mathrm{GL}(n_i, \mathbb{C})/\mathbb{C}^*$, with $\mathbb{C}^*$ embedded diagonally.

We will need a few general results on geometry of representation varieties. First, we recall the following well-known fact (see, for example, [34], Proposition 19.1.4).

Theorem 2.3. *If R is a smooth algebra, then $\operatorname{Rep}(R,\ V)$ is a smooth variety. More generally, let S be a semisimple subalgebra of R, and let $\varrho \in \operatorname{Rep}_S(R, V) \subseteq \operatorname{Rep}(R, V)$. If $\operatorname{Rep}(R, V)$ is smooth at ϱ, then so is $\operatorname{Rep}_S(R, V)$.*

[1] Here, by an affine variety we mean a reduced affine scheme of finite type over $\mathbb{C}$.

Now, given an algebra A, we set $R := T_A\mathbb{D}\mathrm{er}(A)$, see Section 2.1. If A is finitely generated, then so is R, and we consider the variety $\mathrm{Rep}(R, V)$ of representations of R on a vector space V. The following result is proved in [28], Section 5 (see also [50]).

Theorem 2.4. *If A is smooth,* $\mathrm{Rep}(R,\ V)$ *is canonically isomorphic to the cotangent bundle of* $\mathrm{Rep}(A,\ V)$. *In particular,* $\mathrm{Rep}(R,\ V)$ *is smooth.*

Recall that R contains a distinguished element: the derivation $\Delta_A : A \to A^{\otimes 2}$ defined by $x \mapsto x \otimes 1 - 1 \otimes x$. We write

$$\mu : \mathrm{Rep}(R, V) \to \mathrm{End}(V), \qquad \varrho \mapsto \varrho(\Delta_A),$$

for the evaluation map at Δ_A and consider its fibre $F_\xi := \mu^{-1}[\mu(\xi)]$ for some fixed representation $\xi \in \mathrm{Rep}(R, V)$.

Proposition 2.2. *If A is smooth, then F_ξ is smooth at $\varrho \in \mathrm{Rep}(R, V)$ if and only if $\mathrm{End}_R(V_\varrho) \cong \mathbb{C}$.*

Proof. By Theorem 2.4, the variety $\mathrm{Rep}(R, V)$ is smooth. By Lemma 2.1, we also have that $\Delta_A \in [A,\ \mathbb{D}\mathrm{er}(A)]$, and therefore $\mathrm{tr}_V[\varrho(\Delta)] = 0$ for any $\varrho \in \mathrm{Rep}(R, V)$. It follows that

$$\mu : \mathrm{Rep}(R, V) \to \mathrm{End}(V)_0, \tag{2.17}$$

where $\mathrm{End}(V)_0 := \{f \in \mathrm{End}(V) : \mathrm{tr}_V(f) = 0\}$.

To compute the differential of (2.17) we use (2.16) and also $T_\mu\,\mathrm{End}(V)_0 \cong \mathrm{End}(V)_0$. With these identifications, it is easy to see that

$$d\mu_\varrho : \mathrm{Der}(R,\ \mathrm{End}\,V) \to \mathrm{End}(V)_0, \qquad \delta \mapsto \delta(\Delta_A). \tag{2.18}$$

Now, observe that the map $d\mu_\varrho^*$ dual to (2.18) fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{End}(V)_0^* & \xrightarrow{d\mu_\varrho^*} & \mathrm{Der}(R,\ \mathrm{End}\,V)^* \\ \mathrm{tr}_V \big\uparrow & & \big\uparrow i(\mathrm{Tr}\,\hat\omega) \\ \mathrm{End}(V)/\mathbb{C} & \xrightarrow{\mathrm{ad}} & \mathrm{Der}(R,\ \mathrm{End}\,V) \end{array} \tag{2.19}$$

with vertical arrows being *isomorphisms*. Here, tr_V comes from the trace pairing on $\mathrm{End}(V)$ (and hence, it is obviously an isomorphism), and ad is induced by the canonical map, assigning to $f \in \mathrm{End}(V)$ the inner derivation $\mathrm{ad}(f) : a \mapsto [f,\ \varrho(a)]$. The crucial isomorphism $i(\mathrm{Tr}\,\hat\omega)$ is constructed[2] in [28] (see *op. cit.*, the proof of Theorem 6.4.3). Instead of repeating this construction, we simply notice that (2.17) can be viewed as a *moment map* for the natural action of $\mathrm{GL}(V)/\mathbb{C}^*$ on $\mathrm{Rep}(R,\ V)$. The commutativity of (2.19) is then equivalent to the defining equation for moment maps (see [28], (6.4.7)). Now, it remains to note that F_ξ is smooth at ϱ if and only if $d\mu_\varrho$ is surjective. This is equivalent to $d\mu_\varrho^*$ being injective, and hence, in view of (2.19), to $\mathrm{Ker}(\mathrm{ad}) = \{0\}$. Since $\mathrm{Ker}(\mathrm{ad}) \cong \mathrm{End}_R(V)/\mathbb{C}$, this last condition holds if and only if $\mathrm{End}_R(V_\varrho) \cong \mathbb{C}$. The proposition follows. $\square$

3. Calogero–Moser spaces

3.1. Rings of differential operators

Let X be a smooth affine irreducible curve over $\mathbb{C}$, with coordinate ring $\mathcal{O} = \mathcal{O}(X)$, and let $\mathcal{D} = \mathcal{D}(X)$ be the ring of differential operators on X. We recall that $\mathcal{D}$ is a filtered algebra $\mathcal{D} = \bigcup_{k\geq 0} \mathcal{D}_k$, with filtration components $0 \subset \mathcal{D}_0 \subset \cdots \subset \mathcal{D}_{k-1} \subset \mathcal{D}_k \subset \cdots$ defined inductively by

$$\mathcal{D}_k := \{\, D \in \mathrm{End}_{\mathbb{C}}\,\mathcal{O} : [D, f] \in \mathcal{D}_{k-1} \text{ for all } f \in \mathcal{O}\,\}.$$

The elements of $\mathcal{D}_k$ are called *differential operators of order* $\leq k$. In particular, $\mathcal{D}_0 = \mathcal{O}$ consists of multiplication operators by regular functions on X, and $\mathcal{D}_1$ is spanned by $\mathcal{O}$ and the space $\mathrm{Der}(\mathcal{O})$ of derivations of $\mathcal{O}$ (the algebraic vector fields on X). As X is smooth, $\mathcal{O}$ and $\mathrm{Der}(\mathcal{O})$ generate $\mathcal{D}$ as an algebra, and $\mathcal{D}$ shares many properties with the Weyl algebra $A_1(\mathbb{C}) = \mathcal{D}(\mathbb{A}^1)$. For example, like $A_1(\mathbb{C})$, $\mathcal{D}$ is a simple Noetherian domain of global dimension 1; however, unlike $A_1(\mathbb{C})$, $\mathcal{D}$ has a nontrivial K-group.

We write $\bar{\mathcal{D}} := \bigoplus_{k=0}^{\infty} \mathcal{D}_k/\mathcal{D}_{k-1}$ for the associated graded ring of $\mathcal{D}$: this is a commutative algebra isomorphic to the coordinate ring of the cotangent bundle T^*X of X. Given a $\mathcal{D}$-module M with a filtration $\{M_k\}$, we also write $\bar{M} := \bigoplus_{k=0}^{\infty} M_k/M_{k-1}$ for the associated graded $\bar{\mathcal{D}}$-module. Using the standard terminology, we say that $\{M_k\}$ is a *good* filtration if $\bar{M}$ is finitely generated (see, e. g. [18]).

3.2. Stable classification of ideals

Let $K_0(X)$ and $\mathrm{Pic}(X)$ denote the Grothendieck group and the Picard group of X respectively. By definition, $K_0(X)$ is generated by the stable isomorphism classes of (algebraic) vector bundles on X, while the elements of $\mathrm{Pic}(X)$ are the

[2] To avoid confusion, here we use the same notation for this map as in [28].

isomorphism classes of line bundles. As X is affine, we may identify $K_0(X)$ with $K_0(\mathcal{O})$, the Grothendieck group of the ring $\mathcal{O}$, and $\mathrm{Pic}(X)$ with $\mathrm{Pic}(\mathcal{O})$, the ideal class group of $\mathcal{O}$. There are two natural maps $\mathrm{rk} : K_0(X) \to \mathbb{Z}$ and $\det : K_0(X) \to \mathrm{Pic}(X)$ defined by taking the rank and the determinant of a vector bundle respectively. In the case of curves, it is well-known that $\mathrm{rk} \oplus \det : K_0(X) \xrightarrow{\sim} \mathbb{Z} \oplus \mathrm{Pic}(X)$ is a group isomorphism.

Now, let $\mathcal{I}(\mathcal{D})$ denote the set of isomorphism classes of (nonzero) left ideals of $\mathcal{D}$. Unlike Pic in the commutative case, $\mathcal{I}(\mathcal{D})$ carries no natural structure of a group. However, since $\mathcal{D}$ is a hereditary domain, $\mathcal{I}(\mathcal{D})$ can be identified with the space of isomorphism classes of rank 1 projective modules, and there is a natural map relating $\mathcal{I}(\mathcal{D})$ to $\mathrm{Pic}(X)$:

$$\gamma : \mathcal{I}(\mathcal{D}) \to K_0(\mathcal{D}) \xrightarrow{\iota_*^{-1}} K_0(X) \xrightarrow{\det} \mathrm{Pic}(X). \tag{3.1}$$

We recall that $K_0(\mathcal{D})$ is the Grothendieck group of the ring $\mathcal{D}$, whose elements are the stable isomorphism classes of f.g. projective $\mathcal{D}$-modules. The first arrow in (3.1) is the tautological map assigning to the isomorphism class of an ideal in $\mathcal{I}(\mathcal{D})$ its stable class in $K_0(\mathcal{D})$. The second arrow ι_*^{-1} is the inverse of the Quillen isomorphism $\iota_* : K_0(X) \xrightarrow{\sim} K_0(\mathcal{D})$ induced by the inclusion $\iota : \mathcal{O} \hookrightarrow \mathcal{D}$. The role of γ becomes clear from the following result proved in [14].

Theorem 3.1 (*See [14], Proposition 2.1*). *Let M be a projective $\mathcal{D}$-module of rank* 1 *equipped with a good filtration such that $\bar{M}$ is torsion-free. Then*

(a) *there is a unique (up to isomorphism) ideal $\mathcal{I}_M \subseteq \mathcal{O}$, such that $\bar{M}$ is isomorphic to a sub-$\bar{\mathcal{D}}$-module of $\bar{\mathcal{D}}\mathcal{I}_M$ of finite codimension (over $\mathbb{C}$);*

(b) *the class of $\mathcal{I}_M$ in* $\mathrm{Pic}(X)$ *and the codimension $n := \dim_{\mathbb{C}} [\bar{\mathcal{D}}\mathcal{I}_M/\bar{M}]$ are independent of the choice of filtration on M, and we have $\gamma[M] = [\mathcal{I}_M]$;*

(c) *if M and N are two projective $\mathcal{D}$-modules of rank* 1, *then*

$$[M] = [N] \text{ in } K_0(\mathcal{D}) \iff [\mathcal{I}_M] = [\mathcal{I}_N] \text{ in } \mathrm{Pic}(X).$$

Theorem 3.1 shows that the fibres of γ are precisely the stable isomorphism classes of ideals of $\mathcal{D}$: thus, up to isomorphism in $K_0(\mathcal{D})$, the ideals of $\mathcal{D}$ are classified by the elements of $\mathrm{Pic}(X)$. Our goal is to refine this classification by describing the fibres of γ in geometric terms. As we will see in Section 4, each fibre $\gamma^{-1}[\mathcal{I}]$ naturally breaks up into a countable union of affine varieties $\overline{\mathcal{C}}_n(X, \mathcal{I})$. In the next section, we introduce these varieties and study their geometric properties.

3.3. The definition of Calogero–Moser spaces

Given a curve X with a line bundle $\mathcal{I}$, we set $A := \mathcal{O}(X)$ and form the one-point extension $B := A[\mathcal{I}]$. By Lemma 2.2(2), B is a smooth algebra, since A is a smooth and $\mathcal{I}$ is a f. g. projective A-module. As in Section 2.2, we will identify the subalgebra of diagonal matrices in B with $\tilde{A} := A \times \mathbb{C}$, and let $\tilde{i} : B \twoheadrightarrow \tilde{A}$ denote the natural projection, see (2.12). Since $\tilde{i}$ is a nilpotent extension, it is suggestive to think of ‘Spec B’ as a (noncommutative) infinitesimal ‘thickening’ of $\mathrm{Spec}\,\tilde{A} = X \bigsqcup \mathrm{pt}$.

We now prove two auxiliary lemmas. The first lemma implies that B is determined, up to isomorphism, by the class of $\mathcal{I}$ in $\mathrm{Pic}(X)$ and is independent of $\mathcal{I}$ up to Morita equivalence. The second lemma computes the Euler characteristics for representations of B, refining the result of Lemma 2.3.

Lemma 3.1. *For line bundles $\mathcal{I}$ and $\mathcal{J}$, the algebras $A[\mathcal{I}]$ and $A[\mathcal{J}]$ are*

(a) *Morita equivalent;*

(b) *isomorphic if and only if $\mathcal{J} \cong \mathcal{I}^\tau$ for some $\tau \in \mathrm{Aut}(X)$, where $\mathcal{I}^\tau := \tau^*\mathcal{I}$.*

Proof. (a) Given $\mathcal{I}$ and $\mathcal{J}$, we set $\mathcal{L} := \mathrm{Hom}_A(\mathcal{I}, \mathcal{J})$, which is a line bundle on X isomorphic to $\mathcal{J}\,\mathcal{I}^\vee = \mathcal{J} \otimes_A \mathcal{I}^\vee$, where $\mathcal{I}^\vee$ is the dual of $\mathcal{I}$. Then, we extend $\mathcal{L}$ to a line bundle over $\tilde{A}$, letting $\tilde{\mathcal{L}} := \mathcal{L} \times \mathbb{C}$, and define $P := \tilde{\mathcal{L}} \otimes_{\tilde{A}} B$, where $B = A[\mathcal{I}]$. Clearly, P is a f. g. projective B-module. On the other hand, since A is a Dedekind domain, $\mathcal{L} \oplus \mathcal{L} \cong A \oplus \mathcal{L}^2$, where $\mathcal{L}^2 = \mathcal{L} \otimes_A \mathcal{L}$, and hence $\tilde{\mathcal{L}} \oplus \tilde{\mathcal{L}} \cong \tilde{A} \oplus \tilde{\mathcal{L}}^2$. It follows that B is isomorphic to a direct summand of $P \oplus P$, so P is a generator in the category of right B-modules. By Morita Theorem, the ring B is then equivalent to $\mathrm{End}_B(P)$. Now, since $\mathrm{End}_B(P) = \mathrm{Hom}_B(\tilde{\mathcal{L}} \otimes_{\tilde{A}} B, P) \cong \mathrm{Hom}_{\tilde{A}}(\tilde{\mathcal{L}}, P)$ and $P \cong \tilde{\mathcal{L}} \oplus (0, \mathcal{L}\,\mathcal{I})$ as a (right) $\tilde{A}$-module, we have $\mathrm{End}_B(P) \cong A[\mathcal{J}]$.

(b) If $\mathcal{J} \cong \mathcal{I}$, then $A[\mathcal{J}] \cong A[\mathcal{I}]$, by Lemma 2.2(3). Without loss of generality, we may therefore identify $\mathcal{I}$ and $\mathcal{J}$ with ideals in A. Given $\tau \in \mathrm{Aut}(X) = \mathrm{Aut}(A)$, we have then $\mathcal{I}^\tau = \tau^{-1}(\mathcal{I})$, and the natural map $\tau^{-1} : A[\mathcal{I}] \to A[\mathcal{I}^\tau]$ is a required isomorphism. The converse statement is left as an exercise to the reader. $\square$

Lemma 3.2. *For any finite-dimensional B-modules $\boldsymbol{U} = (U, U_\infty)$ and $\boldsymbol{V} = (V, V_\infty)$, we have*

$$\chi_B(\boldsymbol{U}, \boldsymbol{V}) = \dim(U_\infty)\,[\dim(V_\infty) - \dim(V)].$$

Proof. First, observe that $\chi_A(U, V) = 0$ for any pair of finite-dimensional A-modules. Indeed, if U and V have disjoint supports, then $\mathrm{Hom}_A(U, V) = \mathrm{Ext}^1_A(U, V) = 0$, and certainly $\chi_A(U, V) = 0$. By additivity of χ_A, it thus suffices to see that

$\chi_A(U, V) = 0$ for modules U and V supported at one point. If $\mathfrak{m}$ is the maximal ideal of A corresponding to that point, we have $\mathrm{Ext}^i_A(U, V) \cong \mathrm{Ext}^i_{A_\mathfrak{m}}(U, V)$ and $\mathrm{Ext}^i_{A_\mathfrak{m}}(U, V) \cong \mathrm{Tor}^{A_\mathfrak{m}}_i(V^*, U)^*$ for all $i \geq 0$. Thus

$$\chi_A(U, V) = \chi_{A_\mathfrak{m}}(U, V) = \sum (-1)^i \dim_{\mathbb{C}} \mathrm{Tor}^{A_\mathfrak{m}}_i(V^*, U).$$

The vanishing of $\chi_A(U, V)$ follows now from standard intersection theory, since $A_\mathfrak{m}$ is a regular local ring of (Krull) dimension 1, while $\dim_{A_\mathfrak{m}}(U) + \dim_{A_\mathfrak{m}}(V^*) = 0$ (see [48], Chapter V, Part B, Theorem 1).

Identifying $\mathcal{I}$ with an ideal in A and dualizing $0 \to \mathcal{I} \to A \to A/\mathcal{I} \to 0$ by V, we get

$$\dim_{\mathbb{C}} \mathrm{Hom}_A(\mathcal{I}, V) = \dim_{\mathbb{C}}(V) - \chi_A(A/\mathcal{I}, V) = \dim_{\mathbb{C}}(V). \tag{3.2}$$

The result follows now from Lemma 2.3. □

Next, we introduce deformed preprojective algebras over B. For this, we need to compute the trace map $\mathrm{Tr}_B : K_0(B) \to H_0(B)$. Recall that $\mathrm{Tr}_* : K_0 \to H_0$ is a natural transformation of functors on the category of associative algebras, so $\tilde{i} : B \to \tilde{A}$ gives rise to the commutative diagram

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\mathrm{Tr}_B} & H_0(B) \\ \downarrow & & \downarrow \\ K_0(\tilde{A}) & \xrightarrow{\mathrm{Tr}_{\tilde{A}}} & H_0(\tilde{A}) \end{array} \tag{3.3}$$

The two vertical arrows in (3.3) are isomorphisms: the first one is given by Lemma 2.2(6), while the second has the obvious inverse (induced by the inclusion $\tilde{A} \hookrightarrow B$). We will use these isomorphisms to identify $H_0(B) \cong H_0(\tilde{A}) = \tilde{A} \subset B$ and

$$K_0(B) \cong K_0(\tilde{A}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Pic}(X). \tag{3.4}$$

Now, for any commutative algebra (e.g., $\tilde{A}$), the trace map factors through the rank. Hence, with above identifications, Tr_B is completely determined by its values on the first two summands in (3.4), while vanishing on the last. Since $\mathrm{Tr}_B[(1, 0)] = e$ and $\mathrm{Tr}_B[(0, 1)] = e_\infty$, the linear map $\mathrm{Tr}_B : \mathbb{C} \otimes_{\mathbb{Z}} K_0(B) \to H_0(B)$ takes its values in the two-dimensional subspace S of B spanned by e and e_∞. Identifying S with $\mathbb{C}^2$, we may regard the vectors $\boldsymbol{\lambda} := (\lambda,\ \lambda_\infty) = \lambda e + \lambda_\infty e_\infty \in S$ as weights for the family of deformed preprojective algebras associated to B:

$$\Pi^{\boldsymbol{\lambda}}(B) = T_B \mathbb{D}\mathrm{er}(B)/\langle \Delta_B - \boldsymbol{\lambda} \rangle. \tag{3.5}$$

Since A is an integral domain, $\{e,\ e_\infty\}$ is a complete set of primitive orthogonal idempotents in $\Pi^{\boldsymbol{\lambda}}(B)$, and $S = \mathbb{C}e \oplus \mathbb{C}e_\infty$ is the associated semisimple subalgebra of $\Pi^{\boldsymbol{\lambda}}(B)$. Now, for each $\boldsymbol{n} = (n,\ n_\infty) \in \mathbb{N}^2$, we form the variety $\mathrm{Rep}_S(\Pi^{\boldsymbol{\lambda}}(B), \boldsymbol{n})$ of representations of $\Pi^{\boldsymbol{\lambda}}(B)$ of dimension vector $\boldsymbol{n}$ and, with notation of Section 2.3, define

$$\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I}) := \mathrm{Rep}_S(\Pi^{\boldsymbol{\lambda}}(B), \boldsymbol{n}) /\!/ \mathrm{G}_S(\boldsymbol{n}). \tag{3.6}$$

Thus, $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I})$ is an affine scheme, whose (closed) points are in bijection with isomorphism classes of semisimple $\Pi^{\boldsymbol{\lambda}}(B)$-modules of dimension vector $\boldsymbol{n}$.

Lemma 3.3. *For any line bundles $\mathcal{I}$ and $\mathcal{J}$, the schemes $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I})$ and $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{J})$ are isomorphic.*

Proof. By Lemma 3.1, $A[\mathcal{I}]$ and $A[\mathcal{J}]$ are Morita equivalent: the corresponding equivalence is given by

$$\mathtt{Mod}\, A[\mathcal{I}] \xrightarrow{\sim} \mathtt{Mod}\, A[\mathcal{J}], \qquad \boldsymbol{V} \mapsto \tilde{\mathcal{L}} \otimes_{\tilde{A}} \boldsymbol{V}, \tag{3.7}$$

where $\tilde{\mathcal{L}} = \mathcal{J}\mathcal{I}^\vee \times \mathbb{C}$. The functor (3.7) induces an isomorphism of vector spaces $H_0(A[\mathcal{I}]) \xrightarrow{\sim} H_0(A[\mathcal{J}])$ which restricts to the identity on $S \subset \tilde{A}$. By [25], Corollary 5.5, it can then be extended (non-canonically) to a Morita equivalence between $\Pi^{\boldsymbol{\lambda}}(A[\mathcal{I}])$ and $\Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}])$ for any $\boldsymbol{\lambda} \in S$. Now, if $\boldsymbol{V} = (V,\ V_\infty)$ with $\dim_{\mathbb{C}} V < \infty$, we have $\tilde{\mathcal{L}} \otimes_{\tilde{A}} \boldsymbol{V} = (\mathcal{J}\mathcal{I}^\vee \otimes_A V,\ V_\infty)$, so by formula (3.2),

$$\dim_{\mathbb{C}}[\mathcal{J}\mathcal{I}^\vee \otimes_A V] = \dim_{\mathbb{C}} \mathrm{Hom}_A(\mathcal{I}\mathcal{J}^\vee,\ V) = \dim_{\mathbb{C}}(V).$$

This shows that (3.7) preserves dimensions, and its extension to Π^λ induces thus an isomorphism: $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I}) \xrightarrow{\sim} \mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{J})$. □

The next lemma is a generalization of [29], Lemma 4.1: it implies that $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I})$ is empty unless $\boldsymbol{\lambda} \cdot \boldsymbol{n} := \lambda\, n + \lambda_\infty n_\infty$ is zero.

Lemma 3.4. *If $\boldsymbol{\lambda} \cdot \boldsymbol{n} \neq 0$, there are no representations of $\Pi^{\boldsymbol{\lambda}}(B)$ of dimension $\boldsymbol{n}$.*

Proof. If $\boldsymbol{V} = V \oplus V_\infty$ is a $\Pi^{\boldsymbol{\lambda}}(B)$-module of dimension $\boldsymbol{n}$, then e and e_∞ act on $\boldsymbol{V}$ as projectors onto V and V_∞ respectively. The trace of $\boldsymbol{\lambda} = \lambda\, e + \lambda_\infty\, e_\infty \in B$ on $\boldsymbol{V}$ is then equal to $\boldsymbol{\lambda} \cdot \boldsymbol{n}$, and it must be zero, by Proposition 2.1. □

Example 3.1. Let X be the affine line $\mathbb{A}^1$. Any line bundle $\mathcal{I}$ on X is trivial. So, choosing a coordinate on X, we may identify $A \cong \mathbb{C}[x]$ and $\mathcal{I} \cong \mathbb{C}[x]$. The one-point extension of A by $\mathcal{I}$ is then isomorphic to the matrix algebra

$$A[\mathcal{I}] \cong \begin{pmatrix} \mathbb{C}[x] & \mathbb{C}[x] \\ 0 & \mathbb{C} \end{pmatrix},$$

which is, in turn, isomorphic to the path algebra $\mathbb{C}Q$ of the quiver Q consisting of two vertices $\{0, \infty\}$ and two arrows $X : 0 \to 0$ and $\mathrm{v} : \infty \to 0$. In fact, the map sending the vertices 0 and ∞ to the idempotents e and e_∞ and $X \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, $\mathrm{v} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, extends to an algebra isomorphism $\mathbb{C}Q \xrightarrow{\sim} A[\mathcal{I}]$.

Now, let $\bar{Q}$ be the double quiver of Q obtained by adding the reverse arrows $Y := X^*$ and $\mathrm{w} := \mathrm{v}^*$ to the corresponding arrows of Q. Then, for any $\boldsymbol{\lambda} = \lambda\, e + \lambda_\infty e_\infty$, with $(\lambda,\ \lambda_\infty) \in \mathbb{C}^2$, the algebra $\Pi^{\boldsymbol{\lambda}}(Q)$ is isomorphic to the quotient of $\mathbb{C}\bar{Q}$ modulo the relation $[X, Y] + [\mathrm{v}, \mathrm{w}] = \boldsymbol{\lambda}$ (see [25], Theorem 3.1). The ideal generated by this relation is the same as the ideal generated by the elements $[X, Y] + \mathrm{v}\,\mathrm{w} - \lambda\, e$ and $\mathrm{w}\,\mathrm{v} + \lambda_\infty e_\infty$. Thus, the $\Pi^{\boldsymbol{\lambda}}(Q)$-modules can be identified with representations $\boldsymbol{V} = V \oplus V_\infty$ of $\bar{Q}$ in which linear maps $\bar{X}, \bar{Y} \in \mathrm{Hom}(V,\ V)$, $\bar{\mathrm{v}} \in \mathrm{Hom}(V_\infty,\ V)$, $\bar{\mathrm{w}} \in \mathrm{Hom}(V,\ V_\infty)$, given by the action of $X, Y, \mathrm{v}, \mathrm{w}$, satisfy

$$[\bar{X},\ \bar{Y}] + \bar{\mathrm{v}}\,\bar{\mathrm{w}} = \lambda\, \mathrm{Id}_V \quad \text{and} \quad \bar{\mathrm{w}}\,\bar{\mathrm{v}} = -\lambda_\infty\, \mathrm{Id}_{V_\infty}. \tag{3.8}$$

Now, taking $\boldsymbol{\lambda} = (1, -n)$, it is easy to see that all representations of $\Pi^{\boldsymbol{\lambda}}(Q)$ of dimension vector $\boldsymbol{n} = (n,\ 1)$ are simple, and the varieties $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}$ coincide (in this special case) with the classical Calogero–Moser spaces $\mathcal{C}_n$. This coincidence was first noticed by W. Crawley-Boevey (see [26], remark on p. 45). For explanations and further discussion of this example we refer the reader to [13].

Motivated by the above example, we will be interested in representations of $\Pi^{\boldsymbol{\lambda}}(B)$ of dimension $\boldsymbol{n} = (n, 1)$. By Lemma 3.4, such representations may exist only if $\boldsymbol{\lambda} = 0$ or $\boldsymbol{\lambda} = (\lambda, -n\lambda)$, with $\lambda \neq 0$. In this last case, $\Pi^{\boldsymbol{\lambda}}(B)$'s are all isomorphic to each other, so without loss of generality we may assume $\lambda = 1$.

Proposition 3.1. *Let $\boldsymbol{\lambda} = (1, -n)$ and $\boldsymbol{n} = (n, 1)$ with $n \in \mathbb{N}$. Then, for any $\mathcal{I}$, the algebra $\Pi^{\boldsymbol{\lambda}}(B)$ has modules of dimension vector $\boldsymbol{n}$. Every such module is simple.*

Proof. On any B-module of dimension $\boldsymbol{n}$, the element $\boldsymbol{\lambda} = e - n\, e_\infty \in B$ will act with zero trace. Thus, by Proposition 2.1, it suffices to see that there exist *indecomposable* B-modules of this dimension vector. Now, a B-module structure on $\boldsymbol{V} = V \oplus V_\infty$ is determined by an A-module homomorphism $\varphi_V : \mathcal{I} \otimes V_\infty \to V$. If $\boldsymbol{V}$ is decomposable with $\dim(V_\infty) = 1$, then one of its summands must be of the form $\boldsymbol{V}' = V' \oplus V_\infty$, where V' is an A-module summand of V of dimension $< n$. In that case, $\mathrm{Im}(\varphi_V) \subseteq V' \subsetneq V$. Thus, for constructing an indecomposable B-module of dimension $\boldsymbol{n}$, it suffices to construct a torsion A-module V of length n together with a *surjective* A-module map $\varphi : \mathcal{I} \to V$. Geometrically, this can be done as follows.

Identify $\mathcal{I}$ with an ideal in A and fix n distinct points $x_1, x_2, \ldots x_n$ on X outside the zero locus of $\mathcal{I}$. Let $V := A/\mathcal{J}$, where $\mathcal{J}$ is the product of the maximal ideals $\mathfrak{m}_i \subset A$ corresponding to x_i's. Clearly, $A/\mathcal{J} \cong \oplus_{i=1}^n A/\mathfrak{m}_i$ and $\dim_{\mathbb{C}} V = n$. Now, since A is a Dedekind domain and $\mathcal{I} \not\subset \mathfrak{m}_i$ for any i, we have $(A/\mathcal{J}) \otimes_A (A/\mathcal{I}) \cong \oplus_{i=1}^n (A/\mathfrak{m}_i) \otimes_A (A/\mathcal{I}) = 0$ and $\mathrm{Tor}_1^A(A/\mathcal{J}, A/\mathcal{I}) \cong (\mathcal{I} \cap \mathcal{J})/\mathcal{I}\,\mathcal{J} = 0$, so the canonical map $V \otimes_A \mathcal{I} \to V$ is an isomorphism. On the other hand, as V is a cyclic A-module, $\mathcal{I}$ surjects naturally onto $V \otimes_A \mathcal{I}$. Combining $\mathcal{I} \twoheadrightarrow V \otimes_A \mathcal{I} \xrightarrow{\sim} V$, we get the required φ. This proves the first claim of the proposition.

Now, let $\boldsymbol{V} = V \oplus V_\infty$ be any $\Pi^{\boldsymbol{\lambda}}(B)$-module of dimension vector $\boldsymbol{n}$. Let $\boldsymbol{V}'$ be a submodule of $\boldsymbol{V}$ of dimension vector $\boldsymbol{k} = (k,\ k_\infty)$ (say). By Lemma 3.4, we have then $\boldsymbol{\lambda} \cdot \boldsymbol{k} = k - n\,k_\infty = 0$. Since $0 \le k_\infty \le n_\infty = 1$, there are only two possibilities: either $\boldsymbol{k} = 0$ or $\boldsymbol{k} = \boldsymbol{n}$, i.e. $\boldsymbol{V}'$ is either 0 or $\boldsymbol{V}$. Hence $\boldsymbol{V}$ is a simple module. □

Remark. 1. The above argument shows that a B-module $\boldsymbol{V}$ of dimension vector $\boldsymbol{n} = (n, 1)$ lifts to a module over $\Pi^{\boldsymbol{\lambda}}(B)$ if and *only if* it is indecomposable.
2. If $\boldsymbol{V}$ is a B-module with a surjective structure map $\varphi_V : \mathcal{I} \otimes V_\infty \to V$, then $\mathrm{End}_B(\boldsymbol{V}) \subseteq \mathrm{End}(V_\infty)$ and hence $\mathrm{End}_B(\boldsymbol{V}) \cong \mathbb{C}$. (This follows immediately from the diagram (2.11) characterizing B-module homomorphisms.) Hence the modules $\boldsymbol{V}$ constructed in the proof of Proposition 3.1 are actually indecomposables with trivial endomorphism rings.

Definition 1. The variety $\mathcal{C}_{\boldsymbol{n},\boldsymbol{\lambda}}(X, \mathcal{I})$ with $\boldsymbol{\lambda} = (1, -n)$ and $\boldsymbol{n} = (n, 1)$ will be denoted $\mathcal{C}_n(X, \mathcal{I})$ and called the *n-th Calogero–Moser space* of type $(X, \mathcal{I})$.

In view of Proposition 3.1, the varieties $\mathcal{C}_n(X, \mathcal{I})$ parametrize the isomorphism classes of simple $\Pi^{\boldsymbol{\lambda}}(B)$-modules of dimension $\boldsymbol{n} = (n, 1)$; they are non-empty for any $[\mathcal{I}] \in \mathrm{Pic}(X)$ and $n \ge 0$. In the special case, when X is the affine line, $\mathcal{C}_n(X, \mathcal{I})$ coincide with the ordinary Calogero–Moser spaces $\mathcal{C}_n$ (see Example 3.1).

Remark. It follows from our discussion in Section 2.3 (see also [28] and [50]) that the variety $\mathcal{C}_n(X, \mathcal{I})$ can be obtained by symplectic reduction from the cotangent bundle of $\mathrm{Rep}(B, \boldsymbol{n})$. This links our definition of Calogero–Moser spaces to the one proposed by V. Ginzburg (see [8], Definition 1.2).

3.4. Smoothness and irreducibility

One of the main results of [51] says that each $\mathcal{C}_n$ is a smooth affine irreducible variety of dimension $2n$. Theorem 1.1 shows that this holds in general, for an arbitrary curve X. To prove the irreducibility of $\mathcal{C}_n$ we will use a method of Crawley-Boevey [27] which is based on the following simple observation.

Lemma 3.5 ([27], Lemma 6.1)**.** *If X is an equidimensional variety, Y is an irreducible variety and $f : X \to Y$ is a dominant morphism with all fibres irreducible of constant dimension, then X is irreducible.*

Apart from Lemma 3.5, we will need another result from [27]. Recall that if an algebraic group G acts on a variety X, and $Y \subseteq X$ is a G-stable constructible subset of X, then the *number of parameters* of G on Y is defined by

$$\dim_G(Y) := \max_d[\dim(Y \cap X_d) + d - \dim G], \tag{3.9}$$

where X_d is the locally closed subset of X consisting of those points whose stabilizer in G has dimension d. If G acts freely on Y, then Y/G is a variety, and $\dim_G(Y) = \dim Y - \dim G = \dim(Y/G)$. This explains the terminology and justifies the notation $\dim_G(Y)$.

The next lemma relates the dimension of fibres of the restriction map

$$\pi : \mathrm{Rep}_S(\Pi, \boldsymbol{n}) \to \mathrm{Rep}_S(B, \boldsymbol{n})$$

to the number of parameters of $\mathrm{G}_S(\boldsymbol{n})$ on $\mathrm{Rep}_S(B, \boldsymbol{n})$; it is a direct generalization of [27], Lemma 3.4.

Lemma 3.6. *If Y is a $\mathrm{G}_S(\boldsymbol{n})$-stable constructible subset of $\mathrm{Rep}_S(B, \boldsymbol{n})$ contained in the image of π, then*

$$\dim \pi^{-1}(Y) = \dim_G(Y) + n^2 + n, \tag{3.10}$$

where $G := \mathrm{G}_S(\boldsymbol{n})$.

Proof. By Proposition 2.1, the fibre of π over a representation $\boldsymbol{V} \in \mathrm{Rep}_S(B, \boldsymbol{n})$ is isomorphic to a coset of $\mathrm{Ext}^1_B(\boldsymbol{V}, \boldsymbol{V})^*$ and hence, by Lemma 3.2, has dimension

$$\dim \pi^{-1}(\boldsymbol{V}) = \dim_{\mathbb{C}} \mathrm{End}_B(\boldsymbol{V}) - \chi_B(\boldsymbol{V}, \boldsymbol{V}) = \dim_{\mathbb{C}} \mathrm{End}_B(\boldsymbol{V}) + n - 1. \tag{3.11}$$

By partitioning Y we may now assume that all representations in Y have endomorphism ring of dimension r. Then, by (3.11),

$$\dim \pi^{-1}(Y) = \dim Y + r + n - 1. \tag{3.12}$$

On the other hand, each orbit of $\mathrm{G}_S(\boldsymbol{n})$ on Y has dimension

$$\dim \mathrm{G}_S(\boldsymbol{n}) - \dim \mathrm{Aut}_B(\boldsymbol{V})/\mathbb{C}^* = \dim \mathrm{G}_S(\boldsymbol{n}) - r + 1,$$

so since $\mathrm{G}_S(\boldsymbol{n}) \cong [\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})]/\mathbb{C}^* \cong \mathrm{GL}(n, \mathbb{C})$, we have

$$\dim_G(Y) = \dim Y + r - 1 - \dim \mathrm{G}_S(\boldsymbol{n}) = \dim Y + r - 1 - n^2. \tag{3.13}$$

Combining (3.12) and (3.13), we get formula (3.10). $\square$

We are now in position to prove the first main result of this paper, which we stated as Theorem 1.1 in the Introduction.

Proof of Theorem 1.1. The varieties $\mathcal{C}_n(X, \mathcal{I})$ are affine by definition; we need only to show that these are smooth and irreducible. Fix $n \in \mathbb{N}$ and $[\mathcal{I}] \in \mathrm{Pic}(X)$. To simplify the notation we write Π for $\Pi^{\boldsymbol{\lambda}}(B)$ with $\boldsymbol{\lambda} = (1, -n)$ and $G := \mathrm{G}_S(\boldsymbol{n})$. By Proposition 3.1, every Π-module $\boldsymbol{V}$ of dimension $\boldsymbol{n} = (n, 1)$ is simple, so, by Schur Lemma, $\mathrm{End}_\Pi(\boldsymbol{V}) \cong \mathbb{C}$ and $\mathrm{Aut}_\Pi(\boldsymbol{V}) \cong \mathbb{C}^*$. The last isomorphism implies that every point of $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ has trivial stabilizer in G, i. e. the natural action of G on $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ is free. In that case, by Luna's Slice Theorem (see [44], Corollaire III.1.1), the quotient variety $\mathcal{C}_n(X, \mathcal{I}) = \mathrm{Rep}_S(\Pi, \boldsymbol{n})//G$ will be smooth if so is the original variety $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$. Now, to see that $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ is smooth, it suffices to see, by Theorem 2.3, that $\mathrm{Rep}(\Pi, \boldsymbol{n})$ is smooth, and that follows from Proposition 2.2 of Section 2.3. In fact, let $R := T_B\mathbb{D}\mathrm{er}(B)$, and let $\sigma : R \twoheadrightarrow \Pi$ be the canonical projection. Then σ induces the closed embedding of affine varieties $\sigma_* : \mathrm{Rep}(\Pi, \boldsymbol{n}) \hookrightarrow \mathrm{Rep}(R, \boldsymbol{n})$, whose image is a fibre of the evaluation map (2.17). Since for every $\varrho \in \mathrm{Im}(\sigma_*)$, we have $\mathrm{End}_R(\boldsymbol{V}) \cong \mathrm{End}_\Pi(\boldsymbol{V}) \cong \mathbb{C}$, the assumption of Proposition 2.2 holds, and the result follows.

Now, we show that $\mathcal{C}_n(X, \mathcal{I})$ is irreducible of dimension $2n$. For this, we examine first the variety $\mathrm{Rep}_S(B, \boldsymbol{n})$. Since B is smooth, $\mathrm{Rep}_S(B, \boldsymbol{n})$ is smooth, i. e. for every point $\varrho \in \mathrm{Rep}_S(B, \boldsymbol{n})$,

$$\dim_\varrho \mathrm{Rep}_S(B, \boldsymbol{n}) = \dim_{\mathbb{C}} T_\varrho \mathrm{Rep}_S(B, \boldsymbol{n}), \tag{3.14}$$

where $\dim_\varrho$ stands for the local dimension and T_ϱ for the Zariski tangent space of $\mathrm{Rep}_S(B, \boldsymbol{n})$ at ϱ. To evaluate the dimension of this space we identify $T_\varrho \mathrm{Rep}_S(B, \boldsymbol{n}) \cong \mathrm{Der}_S(B, \mathrm{End}\,\boldsymbol{V})$, as in (2.16), and consider the standard exact sequence

$$0 \to \mathrm{End}_B(\boldsymbol{V}) \to \mathrm{End}_S(\boldsymbol{V}) \to \mathrm{Der}_S(B,\ \mathrm{End}\,\boldsymbol{V}) \to H^1(B,\ \mathrm{End}\,\boldsymbol{V}) \to 0.$$

Identifying now terms in this sequence $\mathrm{End}_S(\boldsymbol{V}) \cong \mathtt{Mat}(n, \mathbb{C}) \times \mathbb{C}$, $H^1(B,\ \mathrm{End}\,\boldsymbol{V}) \cong \mathrm{Ext}^1_B(\boldsymbol{V},\ \boldsymbol{V})$, we get

$$\dim_{\mathbb{C}} \mathrm{Der}_S(B,\ \mathrm{End}\,\boldsymbol{V}) = n^2 + 1 - \chi_B(\boldsymbol{V}, \boldsymbol{V}) = n^2 + n. \tag{3.15}$$

Thus $\mathrm{Rep}_S(B, \boldsymbol{n})$ is a smooth equidimensional variety of dimension $n^2 + n$. To see that it is actually irreducible, we apply Lemma 3.5 to the canonical projection

$$f : \mathrm{Rep}_S(B, \boldsymbol{n}) \to \mathrm{Rep}(A, n).$$

In this case, the assumptions of Lemma 3.5 are easy to verify: since X is irreducible, so is clearly $\mathrm{Rep}(A, n)$, and the fibres of f over each $V \in \mathrm{Rep}(A, n)$ can be identified with the vector spaces $\mathrm{Hom}_A(I, V)$ and, hence, are all irreducible of the same dimension n, by formula (3.2). Below we will also need the following property of the quotient map

$$\bar{f} : \mathrm{Rep}_S(B, \boldsymbol{n})/G \to \mathrm{Rep}(A, n)/\mathrm{GL}(n, \mathbb{C})$$

induced by f on the isomorphism classes of representations.

Lemma 3.7. *The fibres of $\bar{f}$ are finite sets.*

Proof. In view of Morita equivalence (see Lemma 3.1), it suffices to consider the trivial case: $\mathcal{I} = A$. For any representation $\varrho : A \to \mathrm{End}(V)$, the fibre of f is then canonically isomorphic to V, and the fibre of $\bar{f}$ is isomorphic to $V/\mathrm{Aut}_A(V)$. Thus we need only to show that V has finitely many orbits under the natural action of $\mathrm{Aut}_A(V)$. For this, it suffices to consider only the case when V is an indecomposable A-module. But in that case $V \cong A/\mathfrak{m}^k$ for some maximal ideal $\mathfrak{m} \subset A$ and $\mathrm{Aut}_A(V) \cong (A/\mathfrak{m}^k)^\times$ acts on V by multiplication, so we have $k+1$ orbits of the form $\mathfrak{m}^i V \setminus \mathfrak{m}^{i+1} V$, where $i = 0, 1, \ldots, k$. □

Next, we consider the restriction map $\pi : \mathrm{Rep}_S(\Pi, \boldsymbol{n}) \to \mathrm{Rep}_S(B, \boldsymbol{n})$. As observed above (*cf.* Remark 1 after Proposition 3.1), the image of π consists exactly of indecomposable modules in $\mathrm{Rep}_S(B, \boldsymbol{n})$, while each (non-empty) fibre $\pi^{-1}(\boldsymbol{V})$ is isomorphic to a coset of $\mathrm{Ext}^1_B(\boldsymbol{V}, \boldsymbol{V})^*$ and is thus irreducible of dimension (3.11).

Now, let $\mathcal{U}$ be the subset of $\mathrm{Rep}_S(B, \boldsymbol{n})$ consisting of modules $\boldsymbol{V}$ with $\mathrm{End}_B(\boldsymbol{V}) \cong \mathbb{C}$. As explained in Remark 2 following Proposition 3.1, this subset is non-empty. By Chevalley's Theorem, the function $\boldsymbol{V} \mapsto \dim_{\mathbb{C}} \mathrm{End}_B(\boldsymbol{V})$ is upper semi-continuous on $\mathrm{Rep}_S(B, \boldsymbol{n})$, i. e.

$$\{\boldsymbol{V} \in \mathrm{Rep}_S(B, \boldsymbol{n}) : \dim_{\mathbb{C}} \mathrm{End}_B(\boldsymbol{V}, \boldsymbol{V}) \geq n\}$$

are closed sets for all $n \in \mathbb{N}$. Hence $\mathcal{U}$ is open in $\mathrm{Rep}_S(B, \boldsymbol{n})$ and therefore dense (since $\mathrm{Rep}_S(B, \boldsymbol{n})$ is irreducible). As $\mathcal{U} \subseteq \mathrm{Im}(\pi)$, this implies that π is a dominant map.

Now, $\pi^{-1}(\mathcal{U})$ is an open subset of $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$, whose local dimension at every point $\varrho \in \pi^{-1}(\mathcal{U})$ equals, by (3.11),

$$\dim_\varrho \pi^{-1}(\mathcal{U}) = \dim \mathcal{U} + \dim \pi^{-1}(\pi(\varrho)) = \dim \mathrm{Rep}_S(B, \boldsymbol{n}) + n = n^2 + 2n.$$

Thus $\pi^{-1}(\mathcal{U})$ is equidimensional and therefore irreducible, by Lemma 3.5. On the other hand, we have

$$\dim \pi^{-1}(\mathrm{Im}\,\pi \setminus \mathcal{U}) < n^2 + 2n. \tag{3.16}$$

To prove (3.16) we take $\boldsymbol{V} \in \mathrm{Rep}_S(B, \boldsymbol{n})$ and think of it as a pair (V, φ) consisting of an A-module V of dimension n and a map $\varphi \in \mathrm{Hom}_A(I, V)$. Suppose that V is semisimple. Then, for $\boldsymbol{V}$ to be indecomposable φ must be surjective (otherwise we could decompose $V = \mathrm{Im}(\varphi) \oplus V'$ as an A-module and $(V', 0)$ would be a direct summand of $\boldsymbol{V}$). By Remark 2 following Proposition 3.1, this implies that $\mathrm{End}_B(\boldsymbol{V}) = \mathbb{C}$, and hence $\boldsymbol{V} \in \mathcal{U}$. Thus the image of $\mathrm{Im}(\pi) \setminus \mathcal{U}$ under the canonical projection $f : \mathrm{Rep}_S(B, \boldsymbol{n}) \to \mathrm{Rep}(A, n)$ is contained in the set $\mathcal{N}$ of nonsemisimple n-dimensional A-modules. This last set is stable under the action of $\mathrm{GL}(n, \mathbb{C})$, and the number of parameters of $\mathrm{GL}(n, \mathbb{C})$ on $\mathcal{N}$ is obviously $< n$. On the other hand, by Lemma 3.7, $\dim_G(Y) = \dim_{\mathrm{GL}_n}[f(Y)]$ for any G-stable constructible subset $Y \subseteq \mathrm{Rep}_S(B, \boldsymbol{n})$. Hence $\dim_G(\mathrm{Im}\,\pi \setminus \mathcal{U}) \leq \dim_{\mathrm{GL}_n}(\mathcal{N}) < n$, and then (3.16) follows from Lemma 3.6.

The inequality (3.16) implies that $\pi^{-1}(\mathcal{U})$ is dense in $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$. Indeed, the variety $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ can be identified with a fibre of the evaluation map $\mu : \mathrm{Rep}_S(R, \boldsymbol{n}) \to \mathrm{End}_S(\boldsymbol{V})_0$, see (2.17), so any irreducible component of $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ has dimension at least

$$\dim \mathrm{Rep}_S(R, \boldsymbol{n}) - \dim \mathrm{End}_S(\boldsymbol{V})_0 = 2(n^2 + n) - n^2 = n^2 + 2n.$$

(Here, we have calculated the dimension of $\mathrm{Rep}_S(R, \boldsymbol{n})$ using the isomorphism of Theorem 2.4.) Thus, in view of (3.16), $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ must coincide with the closure of $\pi^{-1}(\mathcal{U})$, and hence is irreducible of dimension $n^2 + 2n$ since so is $\pi^{-1}(\mathcal{U})$. The irreducibility of $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ now implies the irreducibility of $\mathcal{C}_n(X, \mathcal{I})$, since $\mathcal{C}_n(X, \mathcal{I})$ is a quotient of $\mathrm{Rep}_S(\Pi, \boldsymbol{n})$ by a free action of G. Finally, as $G \cong [\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})]/\mathbb{C}^* \cong \mathrm{GL}(n, \mathbb{C})$, we compute

$$\dim \mathcal{C}_n(X, \mathcal{I}) = \dim \mathrm{Rep}_S(\Pi, \boldsymbol{n}) - \dim G = n^2 + 2n - n^2 = 2n.$$

This completes the proof of the theorem. □

4. The Calogero–Moser correspondence

4.1. Recollement

We begin by clarifying the relation between the algebras $\Pi^\lambda(B)$ and the ring $\mathcal{D}$ of differential operators on X (see also Appendix).

Lemma 4.1. *There is a canonical map $i : \Pi^{\boldsymbol{\lambda}}(B) \to \Pi^1(A)$ which is a surjective pseudo-flat ring homomorphism, with* $\mathrm{Ker}(i) = \langle e_\infty \rangle$.

Proof. By Lemma 2.2(4), the projection $i : B \to A$, see (2.13), is a flat (and hence, pseudo-flat) ring epimorphism. Since B is smooth, by Theorem 2.1, i extends to an algebra map $\Pi^{\boldsymbol{\lambda}}(B) \to \Pi^{i_*(\boldsymbol{\lambda})}(A)$ which is also a pseudo-flat ring epimorphism. Now, since the map i is surjective with $\mathrm{Ker}(i) = \langle e_\infty \rangle$, the Cartesian square (2.2) shows that its extension to $\Pi^{\boldsymbol{\lambda}}(B)$ is also surjective with kernel $\langle e_\infty \rangle$. Finally, with identifications of Section 3.3, it is easy to see that $i_*(\boldsymbol{\lambda}) = 1$. $\square$

Theorem 4.1 (*[25], Theorem 4.7*). *If $A = \mathcal{O}(X)$ is the coordinate ring of a smooth affine curve, then $\Pi^1(A)$ is isomorphic (as a filtered algebra) to $\mathcal{D} = \mathcal{D}(X)$.*

We fix, once and for all, an isomorphism of Theorem 4.1 to identify $\mathcal{D} = \Pi^1(A)$. In combination with Lemma 4.1, this yields an algebra map $i : \Pi^{\boldsymbol{\lambda}}(B) \to \mathcal{D}$. We will use i to relate the (derived) module categories of $\Pi := \Pi^{\boldsymbol{\lambda}}(B)$ and $\mathcal{D}$, as follows (*cf.* [13]). First, we let $U := U^{\boldsymbol{\lambda}}$ denote the endomorphism ring of the projective module $e_\infty \Pi$: this ring can be identified with the associative subalgebra $e_\infty \Pi\, e_\infty$ of Π having e_∞ as an identity element. Next, we introduce six additive functors $(i^*,\ i_*,\ i^!)$ and $(j_!,\ j^*,\ j_*)$ between the module categories of Π, $\mathcal{D}$ and U. We let $i_* : \mathtt{Mod}(\mathcal{D}) \to \mathtt{Mod}(\Pi)$ be the restriction functor associated to i. This functor is fully faithful and has both the right adjoint $i^! := \mathrm{Hom}_\Pi(\mathcal{D},\ -)$ and the left adjoint $i^* := \mathcal{D} \otimes_\Pi -$, with adjunction maps $i^* i_* \simeq \mathrm{Id} \simeq i^! i_*$ being isomorphisms. Now we define $j^* : \mathtt{Mod}(\Pi) \to \mathtt{Mod}(U)$ by $j^*\boldsymbol{V} := e_\infty \boldsymbol{V}$. Since $e_\infty \in \Pi$ is an idempotent, j^* is exact and has also the right and left adjoints: $j_* := \mathrm{Hom}_U(e_\infty \Pi,\ -)$ and $j_! := \Pi e_\infty \otimes_U -$, satisfying $j^* j_* \simeq \mathrm{Id} \simeq j^* j_!$.

The six functors $(i^*,\ i_*,\ i^!)$ and $(j_!,\ j^*,\ j_*)$ defined above extend to the derived categories, and their properties are summarized in Theorem 1.2 from the Introduction.

Proof of Theorem 1.2. Theorem 1.2 follows from general results on recollement of module categories (see [37]) and the following observation which will be proved in Section 5.1 (see Lemma 5.4): the multiplication map $\Pi\, e_\infty \otimes_U e_\infty \Pi \to \Pi$ fits into the exact sequence

$$0 \to \Pi\, e_\infty \otimes_U e_\infty \Pi \to \Pi \xrightarrow{i} \mathcal{D} \to 0 \tag{4.1}$$

which is a projective resolution of $\mathcal{D}$ in the category of (left and right) Π-modules. The existence of (4.1) implies that $\mathcal{D}$ has projective dimension 1 in $\mathtt{Mod}(\Pi)$. Hence $\mathrm{Tor}_n^\Pi(\mathcal{D}, \mathcal{D}) = 0$ for all $n \geq 2$. On the other hand, by Lemma 4.1, i is a pseudo-flat epimorphism, meaning that $\mathrm{Tor}_1^\Pi(\mathcal{D}, \mathcal{D}) = 0$ as well. Theorem 1.2 follows now directly from [37], Corollary 14. $\square$

As another consequence of (4.1), we have

Lemma 4.2. *If $\boldsymbol{V}$ is a finite-dimensional Π-module, then $L_n i^*(\boldsymbol{V}) = 0$ for $n \neq 1$ and*

$$L_1 i^*(\boldsymbol{V}) \cong \mathrm{Ker}\left[\Pi\, e_\infty \otimes_U e_\infty \boldsymbol{V} \xrightarrow{\boldsymbol{\mu}} \boldsymbol{V}\right], \tag{4.2}$$

where $L_n i^*$ denotes the n-th derived functor of i^* and $\boldsymbol{\mu}$ is the natural multiplication-action map.

Proof. Tensoring (4.1) with $\boldsymbol{V}$ yields the exact sequence

$$0 \to \mathrm{Tor}_1^\Pi(\mathcal{D}, \boldsymbol{V}) \to \Pi e_\infty \otimes_U e_\infty \boldsymbol{V} \to \boldsymbol{V} \to \mathcal{D} \otimes_\Pi \boldsymbol{V} \to 0,$$

and isomorphisms $\mathrm{Tor}_n^\Pi(\mathcal{D}, \boldsymbol{V}) \cong \mathrm{Tor}_{n-1}^\Pi(\Pi\, e_\infty \otimes_U e_\infty \Pi, \boldsymbol{V})$ for $n \geq 2$. Since $\Pi\, e_\infty \otimes_U e_\infty \Pi$ is projective (as a right Π-module), the last Tor's vanish. On the other hand, $\dim_{\mathbb{C}} \boldsymbol{V} < \infty$ implies that $\mathcal{D} \otimes_\Pi \boldsymbol{V} = 0$, since $\mathcal{D}$ has no nonzero finite-dimensional modules. The result follows now from the identification $L_n i^*(\boldsymbol{V}) = \mathrm{Tor}_n^\Pi(\mathcal{D},\ \boldsymbol{V})$, $n \geq 0$. $\square$

Remark. Using (1.1), we may define the following 'perverse' t-structure on $\mathcal{D}^b(\mathtt{Mod}\,\Pi)$:

$$^p\mathcal{D}^{\leq 0}(\mathtt{Mod}\,\Pi) := \{\, K^\bullet \in \mathcal{D}^b(\mathtt{Mod}\,\Pi) : j^*K^\bullet \in \mathcal{D}^{\leq 0}(U),\, i^*K^\bullet \in \mathcal{D}^{\leq -1}(\mathcal{D})\,\},$$
$$^p\mathcal{D}^{\geq 0}(\mathtt{Mod}\,\Pi) := \{\, K^\bullet \in \mathcal{D}^b(\mathtt{Mod}\,\Pi) : j^*K^\bullet \in \mathcal{D}^{\geq 0}(U),\, i^!K^\bullet \in \mathcal{D}^{\geq -1}(\mathcal{D})\,\},$$

where $\{\mathcal{D}^{\leq 0}(U),\ \mathcal{D}^{\geq 0}(U)\}$ and $\{\mathcal{D}^{\leq 0}(\mathcal{D}),\ \mathcal{D}^{\geq 0}(\mathcal{D})\}$ denote the standard t-structures on $\mathcal{D}^b(\mathtt{Mod}\,U)$ and $\mathcal{D}^b(\mathtt{Mod}\,\mathcal{D})$ respectively. Lemma 4.2 shows that the 0-complexes $[0 \to \boldsymbol{V} \to 0]$ with $\dim_{\mathbb{C}} \boldsymbol{V} < \infty$ lie in the heart of this t-structure. So we may think of finite-dimensional Π-modules as 'perverse sheaves' with respect to the stratification (1.1). The functor i^* is then an algebraic analogue of the restriction functor of a (perverse) sheaf to a closed subspace.

4.2. *The action of* $\mathrm{Pic}(\mathcal{D})$ *on Calogero–Moser spaces*

We recall some facts about the Picard group $\mathrm{Pic}(\mathcal{D})$ of the algebra $\mathcal{D}$ and its action on the space of ideals $\mathcal{I}(\mathcal{D})$ (see [14]). It is known that $\mathrm{Pic}(\mathcal{D})$ has different descriptions for $X = \mathbb{A}^1$ and other curves (see [20]). Since the case of $\mathbb{A}^1$ is well studied, we will assume that $X \neq \mathbb{A}^1$. Our main theorem (Theorem 4.2) still holds for all curves X, including $\mathbb{A}^1$.

In general, $\mathrm{Pic}(\mathcal{D})$ can be identified with the group of $\mathbb{C}$-linear auto-equivalences of the category $\mathtt{Mod}(\mathcal{D})$, and thus it acts naturally on $\mathcal{I}(\mathcal{D})$ and $K_0(\mathcal{D})$. To be precise, the elements of $\mathrm{Pic}(\mathcal{D})$ are the isomorphism classes $[\mathcal{P}]$ of invertible $\mathcal{D}$-bimodules, and the action of $\mathrm{Pic}(\mathcal{D})$ on $\mathcal{I}(\mathcal{D})$ and $K_0(\mathcal{D})$ is defined by $[M] \mapsto [\mathcal{P} \otimes_{\mathcal{D}} M]$. The action of $\mathrm{Pic}(\mathcal{D})$ on $K_0(\mathcal{D})$ preserves rank and hence restricts to $\mathrm{Pic}(X)$ through the identification $K_0(\mathcal{D}) \cong K_0(X) \cong \mathbb{Z} \oplus \mathrm{Pic}(X)$, see Section 3.2.

Proposition 4.1 (*See [14], Theorem 1.1*). *$\mathrm{Pic}(\mathcal{D})$ acts on $\mathrm{Pic}(X)$ transitively, and the map $\gamma : \mathcal{I}(\mathcal{D}) \to \mathrm{Pic}(X)$ defined by (3.1) is equivariant under this action.*

Explicitly, the action of $\mathrm{Pic}(\mathcal{D})$ on $\mathrm{Pic}(X)$ can be described as follows (*cf.* [14], Proposition 3.1). By [20], Corollary 1.13, every invertible bimodule over $\mathcal{D}$ is isomorphic to $\mathcal{D}\mathcal{L} = \mathcal{D} \otimes_A \mathcal{L}$ as a left module, while the right action of $\mathcal{D}$ is determined by an algebra isomorphism $\varphi : \mathcal{D} \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}}(\mathcal{D}\mathcal{L})$, where $\mathcal{L}$ is a line bundle on X. Following [14], we denote such a bimodule by $(\mathcal{D}\mathcal{L})_\varphi$. Restricting φ to A yields an automorphism of X, and the assignment

$$g : \mathrm{Pic}(\mathcal{D}) \to \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X), \quad [(\mathcal{D}\mathcal{L})_\varphi] \mapsto ([\mathcal{L}], \varphi|_A), \tag{4.3}$$

defines then a group homomorphism. On the other hand, $\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)$ acts on $\mathrm{Pic}(X)$ in the obvious way:

$$([\mathcal{L}], \tau) : [\mathcal{I}] \mapsto [\mathcal{L}\, \tau(\mathcal{I})], \tag{4.4}$$

where $([\mathcal{L}], \tau) \in \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)$ and $[\mathcal{I}] \in \mathrm{Pic}(X)$. Combining (4.3) and (4.4), we get an action of $\mathrm{Pic}(\mathcal{D})$ on $\mathrm{Pic}(X)$ which agrees with the natural action of $\mathrm{Pic}(\mathcal{D})$ on $K_0(\mathcal{D})$.

Now, given a line bundle $\mathcal{I}$ and an invertible bimodule $\mathcal{P} = (\mathcal{D}\mathcal{L})_\varphi$, we define $P := \tilde{\mathcal{L}} \otimes_{\tilde{A}} B_\tau$, where $\tilde{\mathcal{L}} := \mathcal{L} \times \mathbb{C}$, $\tau := \varphi|_A$, and $B_\tau := A[\tau(\mathcal{I})]$. By Lemma 3.1(a), P is a progenerator in the category of right B_τ-modules, with $\mathrm{End}_{B_\tau}(P) \cong A[\mathcal{J}]$, where $\mathcal{J} := \mathcal{L}\, \tau(\mathcal{I})$. Associated to $\mathcal{P}$ is thus the Morita equivalence: $\mathtt{Mod}(B_\tau) \xrightarrow{\sim} \mathtt{Mod}(A[\mathcal{J}])$, $\boldsymbol{V} \mapsto P \otimes_{B_\tau} \boldsymbol{V} \cong \tilde{\mathcal{L}} \otimes_{\tilde{A}} \boldsymbol{V}$.

Next, we extend P to the $\Pi^{\boldsymbol{\lambda}}(B_\tau)$-module $\boldsymbol{P} := P \otimes_{B_\tau} \Pi^{\boldsymbol{\lambda}}(B_\tau) \cong \tilde{\mathcal{L}} \otimes_{\tilde{A}} \Pi^{\boldsymbol{\lambda}}(B_\tau)$ which is clearly a progenerator in the category of right $\Pi^{\boldsymbol{\lambda}}(B_\tau)$-modules. By Lemma 3.1(b), τ defines an isomorphism $B_\tau \xrightarrow{\sim} B$. Since $\tau(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$ for all $\boldsymbol{\lambda} \in S$, this isomorphism canonically extends to deformed preprojective algebras $\tilde{\tau} : \Pi^{\boldsymbol{\lambda}}(B) \xrightarrow{\sim} \Pi^{\boldsymbol{\lambda}}(B_\tau)$ which allows us to regard $\boldsymbol{P}$ as a $\Pi^{\boldsymbol{\lambda}}(B)$-module and identify

$$\mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}) \cong \tilde{\mathcal{F}} \otimes_{\tilde{A}} \Pi^{\boldsymbol{\lambda}}(B) \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee, \quad \mathcal{F} := \mathcal{L}^\tau. \tag{4.5}$$

With this identification, we have the embedding

$$\tilde{\tau}^{-1} : A[\mathcal{J}] \hookrightarrow \mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}), \tag{4.6}$$

and, since $\mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D}) \cong \tilde{\mathcal{F}} \otimes_{\tilde{A}} \mathcal{D} \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee$, the natural map

$$1 \otimes i \otimes 1 : \mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}) \to \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D}). \tag{4.7}$$

On the other hand, $\varphi(\mathcal{D}) = \mathrm{End}_{\mathcal{D}}(\mathcal{D}\mathcal{L}) = \mathcal{L}^\vee \mathcal{D} \mathcal{L}$ implies $\mathcal{D} = \mathcal{L}\, \varphi(\mathcal{D})\, \mathcal{L}^\vee$, so taking the inverse defines an isomorphism $\psi := \varphi^{-1} : \mathcal{D} \to \mathcal{F}\, \mathcal{D}\, \mathcal{F}^\vee = \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D})$. Combining this last isomorphism with (4.7), we get the diagram of algebra maps

$$\begin{tikzcd} \Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}]) \arrow[r, dotted, "\widetilde{\psi}"] \arrow[d, "i"'] & \mathrm{End}_{\Pi^{\boldsymbol{\lambda}}(B)}(\boldsymbol{P}) \arrow[d, "1 \otimes i \otimes 1"] \\ \mathcal{D} \arrow[r, "\psi"] & \mathrm{End}_{\mathcal{D}}(\mathcal{F}\mathcal{D}) \end{tikzcd} \tag{4.8}$$

which obviously commutes when the dotted arrow is restricted to (4.6).

Proposition 4.2. *There is a unique algebra isomorphism $\widetilde{\psi}$ extending (4.6) and making (4.8) a commutative diagram.*

We postpone the proof of Proposition 4.2 until Section 5.5. Meanwhile, we note that the isomorphism $\widetilde{\psi}$ makes $\boldsymbol{P}$ a left $\Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}])$-module and thus a progenerator from $\Pi^{\boldsymbol{\lambda}}(A[\mathcal{I}])$ to $\Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}])$. This assigns to $\mathcal{P} = (\mathcal{D}\mathcal{L})_\varphi$ the Morita equivalence

$$\mathtt{Mod}\, \Pi^{\boldsymbol{\lambda}}(A[\mathcal{I}]) \to \mathtt{Mod}\, \Pi^{\boldsymbol{\lambda}}(A[\mathcal{J}]), \qquad \boldsymbol{V} \mapsto \boldsymbol{P} \otimes_\Pi \boldsymbol{V}$$

which, in turn, induces an isomorphism of representation varieties

$$f_{\mathcal{P}} : \mathcal{C}_n(X, \mathcal{I}) \xrightarrow{\sim} \mathcal{C}_n(X, \mathcal{J}). \tag{4.9}$$

Remark. We warn the reader that (4.9) *depends* on the choice of a specific representative in the class $[\mathcal{P}] \in \mathrm{Pic}(\mathcal{D})$, so, in general, we do not get an action of $\mathrm{Pic}(\mathcal{D})$ on $\bigsqcup_{[\mathcal{I}] \in \mathrm{Pic}(X)} \mathcal{C}_n(X, \mathcal{I})$. However, we will see below (Proposition 4.3) that $f_{\mathcal{P}}$ induces a well-defined action of $\mathrm{Pic}(\mathcal{D})$ on the reduced spaces $\overline{\mathcal{C}}_n(X, \mathcal{I})$.

Next, we describe a natural action of the canonical bundle $\Omega^1 X$ on $\mathcal{C}_n(X, \mathcal{I})$. Recall that the group homomorphism (4.3) is surjective and fits into the exact sequence (see [20], Theorem 1.15)

$$1 \to \Lambda \xrightarrow{\mathtt{dlog}} \Omega^1 X \xrightarrow{c} \mathrm{Pic}(\mathcal{D}) \xrightarrow{g} \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X) \to 1, \tag{4.10}$$

where $\Lambda := A^\times/\mathbb{C}^\times$ is the multiplicative group of (nontrivial) units in A. The maps $\mathtt{dlog}$ and c in (4.10) are defined by

$$\mathtt{dlog}:\ \Lambda \to \Omega^1 X,\ u \mapsto u^{-1}du, \qquad c:\ \Omega^1 X \to \mathrm{Pic}(\mathcal{D}),\ \omega \mapsto [\mathcal{D}_{\bar{\sigma}_\omega}], \tag{4.11}$$

where $\bar{\sigma}_\omega$ is the automorphism of $\mathcal{D}$ acting identically on A and mapping $\partial \in \mathrm{Der}(A)$ to $\omega(\partial) + \partial \in \mathcal{D}_1$. Since the action of $\mathrm{Pic}(\mathcal{D})$ on $\mathrm{Pic}(X)$ factors through g, the image of $\Omega^1 X$ in $\mathrm{Pic}(\mathcal{D})$ under c stabilizes each point of $\mathrm{Pic}(X)$, and therefore, by equivariance of γ, preserves every fibre $\gamma^{-1}[\mathcal{I}] \subseteq \mathfrak{I}(\mathcal{D})$. Thus, writing $\Gamma := \Omega^1(X)/\Lambda$ and identifying Γ with $\mathrm{Im}(c)$, we get an action

$$\Gamma \times \gamma^{-1}[\mathcal{I}] \to \gamma^{-1}[\mathcal{I}], \quad [\mathcal{I}] \in \mathrm{Pic}(X). \tag{4.12}$$

Now, let $(\Omega^1 B)_\natural := \Omega^1(B)/[B,\ \Omega^1 B]$, where $B = A[\mathcal{I}]$. Using the fact that B is smooth, we identify

$$(\Omega^1 B)_\natural \cong B \otimes_{B^e} \Omega^1(B) \cong B \otimes_{B^e} (\Omega^1 B)^{\star\star} \cong \mathrm{Hom}_{B^e}((\Omega^1 B)^\star,\ B),$$

where $(-)^\star$ stands for the dual over B^e. Explicitly, under this identification, $\bar{\omega} = \omega\,(\mathrm{mod}\,[B,\ \Omega^1 B]) \in (\Omega^1 B)_\natural$ corresponds to the homomorphism

$$\hat{\omega}:\ \Omega^1(B)^\star \to B, \qquad \delta \mapsto \mu^\circ\,[\delta(\omega)], \tag{4.13}$$

where $\mu^\circ\ :\ B^e\ \to\ B$ is the opposite multiplication map. The additive group $(\Omega^1 B)_\natural$ acts naturally on $T_B\,(\Omega^1 B)^\star$: for $\bar{\omega} \in (\Omega^1 B)_\natural$, we have an automorphism $\tilde{\sigma}_\omega$ of $T_B\,(\Omega^1 B)^\star$ acting identically on B and mapping

$$(\Omega^1 B)^\star \to B \oplus (\Omega^1 B)^\star \ \hookrightarrow\ T_B\,(\Omega^1 B)^\star, \qquad \delta \mapsto \hat{\omega}(\delta) + \delta.$$

The assignment $\bar{\omega} \mapsto \tilde{\sigma}_\omega$ defines then a group homomorphism

$$\tilde{\sigma}\ :(\Omega^1 B)_\natural \to \mathrm{Aut}_B[T_B\,(\Omega^1 B)^\star]. \tag{4.14}$$

Identifying $\Omega^1 X$ with the group of Kähler differentials of A, we now construct an embedding $\Omega^1 X\ \hookrightarrow\ (\Omega^1 B)_\natural$. For this, we consider the exact sequence

$$0 \to H_1(B) \xrightarrow{\alpha} (\Omega^1 B)_\natural \to B \to H_0(B) \to 0, \tag{4.15}$$

obtained by tensoring $0\ \to\ \Omega^1(B)\ \to\ B^e\ \to\ B\ \to\ 0$ with B, and compose the connecting map α in (4.15) with natural isomorphisms (see [43], Theorem 1.2.15 and Proposition 1.1.10, respectively)

$$H_1(B) \cong H_1(A) \cong \Omega^1 X. \tag{4.16}$$

Now, for any algebra B, we have (see [21], Exercise 19, p. 126)

$$H_1(B) = \mathrm{Tor}_1^{B^e}(B,\ B) \cong \Omega^1(B)\ \cap\ \Omega^1(B)^\circ/\ \Omega^1(B)\cdot \Omega^1(B)^\circ,$$

where $\Omega^1(B)^\circ := \mathrm{Ker}(\mu^\circ)$ and the intersection and product are taken in B^e. Hence, if $\bar{\omega} \in \mathrm{Im}\,\alpha$ in (4.15), then $\Delta_B(\omega) \in \Omega^1(B)\ \cap\ \Omega^1(B)^\circ \subseteq \Omega^1(B)^\circ$, so by (4.13), $\hat{\omega}(\Delta_B) = 0$ and $\tilde{\sigma}_\omega(\Delta_B) = \Delta_B$. Thus, combining (4.14) with (4.15) and (4.16), we may define

$$\sigma:\ \Omega^1 X\ \overset{\alpha}{\hookrightarrow}\ (\Omega^1 B)_\natural \xrightarrow{\tilde{\sigma}} \mathrm{Aut}_B[T_B\,(\Omega^1 B)^\star] \to \mathrm{Aut}_S[\Pi^{\boldsymbol{\lambda}}(B)], \tag{4.17}$$

where the last map is induced by the algebra projection: $T_B\,(\Omega^1 B)^\star\ \twoheadrightarrow\ \Pi^{\boldsymbol{\lambda}}(B)$. An explicit description of $\tilde{\sigma}$ will be given in Section 5.4 (see Lemma 5.10).

Now, the group $\mathrm{Aut}_S[\Pi^{\boldsymbol{\lambda}}(B)]$ acts on $\mathrm{Rep}_S(\Pi^{\boldsymbol{\lambda}}(B), \boldsymbol{n})$ in the natural way: if $\varrho\ :\ \Pi^{\boldsymbol{\lambda}}(B)\ \to\ \mathrm{End}(\boldsymbol{V})$ represents a point in $\mathrm{Rep}_S(\Pi^{\boldsymbol{\lambda}}(B), \boldsymbol{n})$, then $\sigma.\varrho = \varrho\,\sigma^{-1}$ for $\sigma \in \mathrm{Aut}_S[\Pi^{\boldsymbol{\lambda}}(B)]$. Clearly, this commutes with the $\mathrm{G}_S(\boldsymbol{n})$-action on $\mathrm{Rep}_S(\Pi^{\boldsymbol{\lambda}}(B), \boldsymbol{n})$, and hence induces an action of $\mathrm{Aut}_S[\Pi^{\boldsymbol{\lambda}}(B)]$ on $\mathcal{C}_n(X, \mathcal{I})$. Restricting this last action to $\Omega^1 X$ via (4.17), we define

$$\sigma^*:\ \Omega^1 X \to \mathrm{Aut}\,[\mathcal{C}_n(X, \mathcal{I})], \qquad \omega \mapsto [\sigma^*_\omega:\ \varrho \mapsto \varrho\,\sigma_\omega^{-1}]. \tag{4.18}$$

Equivalently, σ^*_ω is defined on $\mathcal{C}_n(X, \mathcal{I})$ by twisting the structure of $\Pi^{\boldsymbol{\lambda}}(B)$-modules by σ_ω^{-1}, i. e. $[\boldsymbol{V}] \mapsto [\boldsymbol{V}^{\sigma_\omega^{-1}}]$. Restricting (4.18) further to Λ, via (4.11), we define the quotient varieties

$$\overline{\mathcal{C}}_n(X, \mathcal{I}) := \mathcal{C}_n(X, \mathcal{I})/\Lambda. \tag{4.19}$$

These varieties come equipped with the induced action of the group $\Gamma = \Omega^1(X)/\Lambda$.

Proposition 4.3. (1) *The action* (4.18) *agrees with* (4.9)*: if $\mathcal{P} = \mathcal{D}_{\bar{\sigma}_\omega}$, then $f_{\mathcal{P}} = \sigma^*_\omega$ for all $\omega \in \Omega^1 X$.*

(2) *The map* (4.9) *induces an isomorphism of quotient varieties $\bar{f}_{\mathcal{P}}\ :\ \overline{\mathcal{C}}_n(X, \mathcal{I}) \xrightarrow{\sim} \overline{\mathcal{C}}_n(X, \mathcal{J})$ which depends only on the class of $\mathcal{P}$ in* $\mathrm{Pic}(\mathcal{D})$.

We will prove Proposition 4.3 in Section 5.5. Here, we make only two remarks.

1. It follows from Proposition 4.3 that the action of Λ on $\mathcal{C}_n(X, \mathcal{I})$ defined above coincides with the natural action of $\mathrm{Aut}(\mathcal{I}) = A^{\times}$, so $\overline{\mathcal{C}}_n(X, \mathcal{I})$ depends only on the class of $\mathcal{I}$ in $\mathrm{Pic}(X)$ and the definition (4.19) agrees with the one given in the introduction.

2. For each $n \geq 0$, let $\overline{\mathcal{C}}_n(X)$ denote the disjoint union of $\overline{\mathcal{C}}_n(X, \mathcal{I})$ over all $[\mathcal{I}] \in \mathrm{Pic}(X)$. By part (2) of Proposition 4.3, the assignment $[\mathcal{P}] \mapsto \bar{f}_{\mathcal{P}}$ defines then an action of $\mathrm{Pic}(\mathcal{D})$ on $\overline{\mathcal{C}}_n(X)$, and part (1) says that this action restricts to the action of Γ on each individual fibre $\overline{\mathcal{C}}_n(X, \mathcal{I})$, i. e. $\bar{f}_{c(\omega)} = \bar{\sigma}^*_{\omega}$ for all $\omega \in \Gamma$.

4.3. The main theorem

We may now put pieces together and state the main result of the present paper. We recall the functor $L_1 i^* = \mathrm{Tor}_1^{\Pi}(\mathcal{D}, -) : \mathtt{Mod}(\Pi) \to \mathtt{Mod}(\mathcal{D})$ associated to $i : \Pi \to \mathcal{D}$: when restricted to finite-dimensional representations, this functor is given by (4.2).

Theorem 4.2. *Let X be a smooth affine irreducible curve over $\mathbb{C}$.*

(a) *For each $n \geq 0$ and $[\mathcal{I}] \in \mathrm{Pic}(X)$, the functor (4.2) induces an injective map*

$$\omega_n : \overline{\mathcal{C}}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}]$$

which is equivariant under the action of the group Γ.

(b) *Amalgamating the maps ω_n for all $n \geq 0$ gives a bijective correspondence*

$$\omega : \bigsqcup_{n \geq 0} \overline{\mathcal{C}}_n(X, \mathcal{I}) \xrightarrow{\sim} \gamma^{-1}[\mathcal{I}].$$

(c) *For any $[\mathcal{I}]$ and $[\mathcal{J}]$ in $\mathrm{Pic}(X)$ and for any $[\mathcal{P}] \in \mathrm{Pic}(\mathcal{D})$, such that $[\mathcal{P}] \cdot [\mathcal{I}] = [\mathcal{J}]$, there is a commutative diagram:*

$$\begin{tikzcd} \overline{\mathcal{C}}_n(X, \mathcal{I}) \arrow[r, "\bar{f}_{\mathcal{P}}"] \arrow[d, "\omega_n"'] & \overline{\mathcal{C}}_n(X, \mathcal{J}) \arrow[d, "\omega_n"] \\ \gamma^{-1}[\mathcal{I}] \arrow[r, "{[\mathcal{P}]}"] & \gamma^{-1}[\mathcal{J}] \end{tikzcd} \tag{4.20}$$

where $\bar{f}_{\mathcal{P}}$ is an isomorphism induced by (4.9).

Remark. For technical reasons, we assumed above that $X \neq \mathbb{A}^1$. Theorem 4.2 holds true, however, in general: if $X = \mathbb{A}^1$, the map ω induced by i^* agrees with the Calogero–Moser map constructed in [15,16] (see [13], Theorem 1). In this case, the ring $\mathcal{D}$ is isomorphic to the Weyl algebra $A_1(\mathbb{C})$, $\mathrm{Pic}(\mathcal{D})$ is isomorphic to the automorphism group $\mathrm{Aut}(A_1)$ of A_1 (see [49]) and Γ corresponds to the subgroup of KP flows in $\mathrm{Aut}(A_1)$ (see [15]). Since $\mathrm{Pic}(\mathbb{A}^1)$ is trivial, the last part of Theorem 4.2 implies the equivariance of ω under the action of $\mathrm{Aut}(A_1)$.

5. Proof of the main theorem

We proceed in four steps. First, we show that the functor (4.2) induces well-defined maps $\tilde{\omega}_n : \mathcal{C}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}]$, one for each integer $n \geq 0$. Second, we prove that every class $[M] \in \gamma^{-1}[\mathcal{I}]$ is contained in the image of $\tilde{\omega}_n$ for some n (which is uniquely determined by $[M]$). Third, we check that $\tilde{\omega}_n$ factors through the action of Λ on $\mathcal{C}_n(X, \mathcal{I})$ and prove that the induced map $\omega_n : \overline{\mathcal{C}}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}]$ is injective and Γ-equivariant. Finally, we prove Propositions 4.2 and 4.3 of Section 4.2, and show that the diagram (4.20) in Theorem 4.2 is commutative.

We begin by describing the algebras $\Pi^{\boldsymbol{\lambda}}(B)$ in terms of generators and relations.

5.1. The structure of $\Pi^{\boldsymbol{\lambda}}(B)$

Recall that, for each $\boldsymbol{\lambda} \in S$, we defined these algebras by formula (3.5), where $\Delta_B \in \mathbb{D}\mathrm{er}(B)$ is the distinguished derivation mapping $x \mapsto x \otimes 1 - 1 \otimes x$. Now, $\mathbb{D}\mathrm{er}(B)$ contains a canonical sub-bimodule $\mathbb{D}\mathrm{er}_S(B)$, consisting of S-linear derivations. We write $\Delta_{B,S} : B \to B \otimes B$ for the inner derivation $x \mapsto \mathrm{ad}_{\mathbf{e}}(x)$, with $\mathbf{e} := e \otimes e + e_{\infty} \otimes e_{\infty} \in B \otimes B$. It is easy to see that $\Delta_{B,S}(x) = 0$ for all $x \in S$, so $\Delta_{B,S} \in \mathbb{D}\mathrm{er}_S(B)$.

Lemma 5.1. *For any $\boldsymbol{\lambda} \in S$, there is a canonical algebra isomorphism*

$$\Pi^{\boldsymbol{\lambda}}(B) \cong T_B \mathbb{D}\mathrm{er}_S(B) / \langle \Delta_{B,S} - \boldsymbol{\lambda} \rangle.$$

Proof. By universal property, the natural embedding of bimodules $\mathbb{D}\mathrm{er}_S(B) \hookrightarrow \mathbb{D}\mathrm{er}(B)$ extends to their tensor algebras. Combined with canonical projection, this yields the algebra map $\phi : T_B \mathbb{D}\mathrm{er}_S(B) \hookrightarrow T_B \mathbb{D}\mathrm{er}(B) \twoheadrightarrow \Pi^{\boldsymbol{\lambda}}(B)$. An easy calculation shows that $\Delta_{B,S} = e\,\Delta_B\,e + e_\infty\,\Delta_B\,e_\infty$ in $\mathbb{D}\mathrm{er}(B)$. So $\Delta_{B,S} - \boldsymbol{\lambda} = e\,(\Delta_B - \boldsymbol{\lambda})\,e + e_\infty(\Delta_B - \boldsymbol{\lambda})\,e_\infty$ belongs to the ideal $\langle \Delta_B - \boldsymbol{\lambda}\rangle \subseteq T_B\mathbb{D}\mathrm{er}(B)$, and hence ϕ vanishes on $\Delta_{B,S} - \boldsymbol{\lambda}$, inducing an algebra map

$$T_B\mathbb{D}\mathrm{er}_S(B)/\langle \Delta_{B,S} - \boldsymbol{\lambda}\rangle \to \Pi^{\boldsymbol{\lambda}}(B). \tag{5.1}$$

We leave as an exercise to the reader to check that (5.1) is an isomorphism. $\square$

By Lemma 5.1, the structure of $\Pi^{\boldsymbol{\lambda}}(B)$ is determined by the bimodule $\mathbb{D}\mathrm{er}_S(B)$. We now describe this bimodule explicitly, in terms of A, $\mathcal{I}$ and the dual module $\mathcal{I}^\vee = \mathrm{Hom}_A(\mathcal{I}, A)$. To fix notation we begin with a few fairly obvious remarks on bimodules over one-point extensions.

A bimodule Ξ over B is characterized by the following data: an A-bimodule T, a left A-module U, a right A-module V and a $\mathbb{C}$-vector space W given together with three A-module homomorphisms $f_1 : \mathcal{I}\otimes V \to T, f_2 : \mathcal{I}\otimes W \to U, g_1 : T\otimes_A \mathcal{I} \to U$ and a $\mathbb{C}$-linear map $g_2 : V\otimes_A \mathcal{I} \to W$ which fit into the commutative diagram

$$\begin{tikzcd} \mathcal{I}\otimes V\otimes_A \mathcal{I} \arrow[r, "\mathrm{Id}\otimes g_2"] \arrow[d, "f_1\otimes_A \mathrm{Id}"'] & \mathcal{I}\otimes W \arrow[d, "f_2"] \\ T\otimes_A \mathcal{I} \arrow[r, "g_1"] & U \end{tikzcd} \tag{5.2}$$

These data can be conveniently organized by using the matrix notation

$$\Xi = \begin{pmatrix} T & U \\ V & W \end{pmatrix}, \tag{5.3}$$

with understanding that B acts on Λ by the usual matrix multiplication, via the maps f_1, f_2, g_1 and g_2. Note that the components of T are determined by

$$T = e\,\Xi\,e, \qquad U = e\,\Xi\,e_\infty, \qquad V = e_\infty\Xi\,e, \qquad W = e_\infty\Xi\,e_\infty, \tag{5.4}$$

and the commutativity of (5.2) ensures the associativity of the action of B. For example, the free bimodule $B\otimes B$ can be decomposed into a direct sum of four bimodules, each of which is easy to identify using (5.4):

$$Be\otimes eB \cong \begin{pmatrix} A\otimes A & A\otimes \mathcal{I} \\ 0 & 0 \end{pmatrix}, \qquad Be\otimes e_\infty B \cong \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \tag{5.5}$$

$$Be_\infty\otimes eB \cong \begin{pmatrix} \mathcal{I}\otimes A & \mathcal{I}\otimes \mathcal{I} \\ A & \mathcal{I} \end{pmatrix}, \qquad Be_\infty\otimes e_\infty B \cong \begin{pmatrix} 0 & \mathcal{I} \\ 0 & \mathbb{C} \end{pmatrix}. \tag{5.6}$$

With this notation, the bimodule $\mathbb{D}\mathrm{er}_S(B)$ can be described as follows.

Lemma 5.2. *There is an isomorphism of B-bimodules*

$$\mathbb{D}\mathrm{er}_S(B) \cong \begin{pmatrix} \mathbb{D}\mathrm{er}(A) & \mathrm{Der}(A, \mathcal{I}\otimes A) \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} \mathcal{I}\otimes \mathcal{I}^\vee & \mathcal{I}\otimes A \\ \mathcal{I}^\vee & A \end{pmatrix},$$

with $\Delta_{B,S}$ corresponding to the element

$$\left[\begin{pmatrix} \Delta_A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\sum_i \mathrm{v}_i\otimes \mathrm{w}_i & 0 \\ 0 & 1 \end{pmatrix}\right], \tag{5.7}$$

where $\{\mathrm{v}_i\}$ and $\{\mathrm{w}_i\}$ are dual bases for the projective A-modules $\mathcal{I}$ and $\mathcal{I}^\vee$.

Proof. With identifications (5.5) and (5.6), it is easy to show that

$$\Omega^1_S B \cong \begin{pmatrix} \Omega^1 A & \Omega^1 A\otimes_A \mathcal{I} \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} 0 & \mathcal{I} \\ 0 & 0 \end{pmatrix}, \tag{5.8}$$

with inclusion $\Omega^1_S B \hookrightarrow B\otimes_S B = (Be\otimes eB) \oplus (Be_\infty\otimes e_\infty B)$ corresponding to the map (i, s), where i is the natural embedding of the first summand of $\Omega^1_S B$ into $Be\otimes eB$, see (5.5), and s is a B-bimodule section $\tilde{\mathcal{I}} \to B\otimes_S B$ given by

$$s : \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mapsto \left[\begin{pmatrix} 0 & -\sum_i \mathrm{w}_i(b)\otimes \mathrm{v}_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right], \quad b\in \mathcal{I}.$$

Note that i is canonical, while s depends on the choice of dual bases for $\mathcal{I}$ and $\mathcal{I}^\vee$.

To describe $\mathbb{D}\mathrm{er}_S(B)$ we now dualize (5.8) and use $\mathbb{D}\mathrm{er}_S(B) = \mathrm{Hom}_{B^e}(\Omega^1_S B, B^{\otimes 2})$, which after trivial identifications yields

$$\mathbb{D}\mathrm{er}_S(B) \cong \begin{pmatrix} \mathbb{D}\mathrm{er}(A) & \mathrm{Der}(A, \mathcal{I}\otimes A) \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} \mathcal{I}\otimes \mathcal{I}^\vee & \mathcal{I}\otimes \mathrm{End}_A(\mathcal{I}) \\ \mathcal{I}^\vee & \mathrm{End}_A(\mathcal{I}) \end{pmatrix}. \tag{5.9}$$

Since A is commutative and $\mathcal{I}$ is a rank 1 projective, $\mathrm{End}_A(\mathcal{I}) = A$, so (5.9) is the required decomposition. With identification (5.8), the element (5.7) corresponds to the embedding $(i, s) : \Omega^1_S B \to (Be\otimes eB) \oplus (Be_\infty\otimes e_\infty B) \hookrightarrow B\otimes B$, which, in turn, corresponds under (5.9) to the element $\Delta_{B,S} \in \mathbb{D}\mathrm{er}_S(B)$. $\square$

Now, using the isomorphism of Lemma 5.1, we identify $\Pi^{\boldsymbol{\lambda}}(B)$ as a quotient of the tensor algebra of the bimodule $\mathbb{D}\mathrm{er}_S(B)$. Keeping the notation of Lemma 5.2, we then have

Proposition 5.1. *The algebra $\Pi^{\boldsymbol{\lambda}}(B)$ is generated by (the images of) the following elements*

$$\hat{a} := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\mathrm{v}}_i := \begin{pmatrix} 0 & \mathrm{v}_i \\ 0 & 0 \end{pmatrix}, \qquad \hat{d} := \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\mathrm{w}}_i := \begin{pmatrix} 0 & 0 \\ \mathrm{w}_i & 0 \end{pmatrix},$$

where $\hat{a}$, $\hat{\mathrm{v}}_i \in B$ and $\hat{d}$, $\hat{\mathrm{w}}_i \in \mathbb{D}\mathrm{er}_S(B)$ with $d \in \mathbb{D}\mathrm{er}(A)$. Apart from the obvious relations induced by matrix multiplication, these elements satisfy

$$\hat{\Delta}_A - \sum_{i=1}^{N} \hat{\mathrm{v}}_i \cdot \hat{\mathrm{w}}_i = \lambda\, e, \qquad \sum_{i=1}^{N} \hat{\mathrm{w}}_i \cdot \hat{\mathrm{v}}_i = \lambda_\infty\, e_\infty, \tag{5.10}$$

where '·' denotes the action of B on the bimodule $\mathbb{D}\mathrm{er}_S(B)$.

Proof. By Lemma 2.2, the matrices $\{\hat{a}\}$ and $\{\hat{\mathrm{v}}_i\}$ generate the algebra B, while $\{\hat{d}\}$ and $\{\hat{\mathrm{w}}_i\}$ generate the first and the second bimodule summand of (5.9) respectively. All together they thus generate the tensor algebra. Now, the ideal $\langle \Delta_{B,S} - \boldsymbol{\lambda}\rangle$ in $\Pi^{\boldsymbol{\lambda}}(B)$ is generated by $e\,(\Delta_{B,S} - \boldsymbol{\lambda})\,e = e\,\Delta_{B,S}\,e - \lambda\,e$ and $e_\infty\,(\Delta_{B,S} - \boldsymbol{\lambda})\,e_\infty = e_\infty\,\Delta_{B,S}\,e_\infty - \lambda_\infty\,e_\infty$, since the sum of these elements is equal to $\Delta_{B,S} - \boldsymbol{\lambda}$. With identification of Lemma 5.2, we then have

$$e\,\Delta_{B,S}\,e = \hat{\Delta}_A - \sum_{i=1}^{N} \hat{\mathrm{v}}_i \cdot \hat{\mathrm{w}}_i, \qquad e_\infty\,\Delta_{B,S}\,e_\infty = \sum_{i=1}^{N} \hat{\mathrm{w}}_i \cdot \hat{\mathrm{v}}_i,$$

whence the relations (5.10). □

Using Proposition 5.1, we now prove two technical results which we use repeatedly in this paper (Lemma 5.4 below was already mentioned in Section 4.1).

Lemma 5.3. *If $\lambda_\infty \neq 0$, the algebra $\Pi^{\boldsymbol{\lambda}}(B)$ is Morita equivalent to $e\,\Pi^{\boldsymbol{\lambda}}(B)\,e$.*

Proof. By standard Morita theory, it suffices to show that $\Pi^{\boldsymbol{\lambda}}(B)\,e\,\Pi^{\boldsymbol{\lambda}}(B) = \Pi^{\boldsymbol{\lambda}}(B)$. This last identity holds in $\Pi^{\boldsymbol{\lambda}}(B)$ if $1 \in \Pi^{\boldsymbol{\lambda}}(B)\,e\,\Pi^{\boldsymbol{\lambda}}(B)$, or equivalently, if $e_\infty \in \Pi^{\boldsymbol{\lambda}}(B)\,e\,\Pi^{\boldsymbol{\lambda}}(B)$, since $e + e_\infty = 1$. But if $\lambda_\infty \neq 0$, the second relation of (5.10) can be written as

$$e_\infty = \frac{1}{\lambda_\infty} \sum_{i=1}^{N} \hat{\mathrm{w}}_i \cdot \hat{\mathrm{v}}_i = \frac{1}{\lambda_\infty} \sum_{i=1}^{N} \hat{\mathrm{w}}_i \cdot e \cdot \hat{\mathrm{v}}_i, \tag{5.11}$$

whence the result. □

In the next lemma, we return to the special case $\boldsymbol{\lambda} = (1, -\boldsymbol{n})$ and use the abbreviation $\Pi := \Pi^{\boldsymbol{\lambda}}(B)$. We also recall the map $i : \Pi \to \mathcal{D}$ given by Lemma 4.1.

Lemma 5.4. *The multiplication map $\Pi\, e_\infty \otimes_U e_\infty \Pi \to \Pi$ gives a projective resolution of $\mathcal{D}$ in the category of (left and right) Π-modules, see (4.1).*

Proof. We identify $A \cong eBe \subset e\Pi e$ via $a \mapsto \hat{a}$ and $\mathcal{I} \cong eBe_\infty \subset e\Pi e_\infty$ via $\mathrm{v} \mapsto \hat{\mathrm{v}}$. Then tensoring i with $\mathcal{I}$ yields $e\Pi e \otimes_{eBe} eBe_\infty \to \mathcal{D} \otimes_A \mathcal{I} \cong \mathcal{D}\mathcal{I}$. Since eBe_∞ is a projective eBe-module, the multiplication map $e\Pi e \otimes_{eBe} eBe_\infty \to e\Pi e_\infty$ is an isomorphism onto $e\Pi eBe_\infty \subseteq e\Pi e_\infty$. On the other hand, $e\Pi e_\infty \subseteq e\Pi eBe_\infty$, by (5.11). Thus, identifying $e\Pi e \otimes_{eBe} eBe_\infty \cong e\Pi e_\infty$, we get a surjective map of left A-modules: $e\Pi e_\infty \twoheadrightarrow \mathcal{D}\mathcal{I}$. Since $\mathcal{D}\mathcal{I}$ is projective over A, the last map has an A-linear section which we denote by $s : \mathcal{D}\mathcal{I} \hookrightarrow e\Pi e_\infty$. Now, using this section, we consider

$$\mathcal{D}\mathcal{I} \otimes e_\infty \Pi e \xrightarrow{s \otimes 1} e\Pi e_\infty \otimes e_\infty \Pi e \twoheadrightarrow e\,\Pi\, e_\infty \otimes_U e_\infty \Pi e \tag{5.12}$$

which is a homomorphism of right $e\Pi e$-modules. Since $e\Pi e_\infty = e\Pi eBe_\infty = \sum_i e\Pi e\,\hat{\mathrm{v}}_i$, the map (5.12) is surjective. On the other hand, using filtrations, it is easy to show that the composition of (5.12) with multiplication map $e\Pi e_\infty \otimes_U e_\infty \Pi e \to e\Pi e$ is injective. Hence (5.12) is injective and therefore an isomorphism. This implies that $e\Pi e_\infty \otimes_U e_\infty \Pi e$ is a right projective $e\Pi e$-module (since obviously so is $\mathcal{D}\mathcal{I} \otimes e_\infty \Pi e$), and $0 \to e\,\Pi\, e_\infty \otimes_U e_\infty \Pi e \to e\Pi e \xrightarrow{i} \mathcal{D} \to 0$ is an exact sequence of $e\Pi e$-modules. By Morita equivalence of Lemma 5.3, the complex $0 \to \Pi\, e_\infty \otimes_U e_\infty \Pi \to \Pi \to 0$ is then a projective resolution of $\mathcal{D}$ in the category of right Π-modules. A similar argument shows that this complex is also a projective resolution of $\mathcal{D}$ as a left Π-module. □

5.2. The map ω is well-defined

We show that the functor (4.2) maps the Π-modules of dimension vector $\boldsymbol{n} = (n, 1)$ to rank 1 torsion-free $\mathcal{D}$-modules M with $\gamma[M] = [\mathcal{I}]$.

Let $\boldsymbol{V}$ be a Π-module of dimension vector $\boldsymbol{n}$, and let $\boldsymbol{L} := \Pi e_\infty \otimes_U e_\infty \boldsymbol{V}$. Write $V := e\boldsymbol{V}$, $V_\infty := e_\infty \boldsymbol{V}$, and similarly $L := e\boldsymbol{L}$, $L_\infty := e_\infty \boldsymbol{L}$ (so that $\dim V_\infty = \dim L_\infty = 1$). Fix a vector $\xi \neq 0$ in V_∞ and define a character $\varepsilon : U \to \mathbb{C}$ by $u.\xi = \varepsilon(u)\xi$ for all $u \in U$. Note that ε does not depend on the choice of ξ and uniquely determines $\boldsymbol{L}$ (and $\boldsymbol{V}$). In fact, we have the isomorphism of Π-modules

$$\Pi\, e_\infty \Big/ \sum_{u \in U} \Pi\, e_\infty (u - \varepsilon(u)) \xrightarrow{\sim} \boldsymbol{L}\,, \quad [e_\infty] \mapsto e_\infty \otimes \xi. \tag{5.13}$$

Now, under the equivalence of Lemma 5.3, $\boldsymbol{\mu} : \boldsymbol{L} \to \boldsymbol{V}$ transforms to a homomorphism of $e\Pi e$-modules $\mu : L \to V$. Since $e_\infty(\operatorname{Ker} \boldsymbol{\mu}) = 0$, we have $\operatorname{Ker} \boldsymbol{\mu} = e\,(\operatorname{Ker} \boldsymbol{\mu}) = \operatorname{Ker} \mu$. Thus $i^*(\boldsymbol{V}) \cong \operatorname{Ker} \mu$, which is naturally an isomorphism of $\mathcal{D}$-modules via $i|_{e\Pi e} : e\Pi e \to \mathcal{D}$.

Next, we set $R := T_A \mathbb{D}\mathrm{er}(A)$ and define the algebra map

$$R \to e\Pi e, \qquad a \mapsto \hat{a}, \qquad d \mapsto \hat{d}, \tag{5.14}$$

where $a \in A$ and $d \in \mathbb{D}\mathrm{er}(A)$. Extending the notation of Proposition 5.1, we will write $\hat{r} \in e\Pi e$ for the image of any element $r \in R$ under (5.14). Note that the natural projection $R \twoheadrightarrow \Pi^1(A) = \mathcal{D}$ factors through (5.14), and the corresponding quotient map is $i|_{e\Pi e}$. The following observation is an easy consequence of (5.13) and Lemma 5.4.

Lemma 5.5. *There is an isomorphism of R-modules*

$$L \cong R\,\mathcal{I} \Big/ \sum_{i=1}^{N} \sum_{r \in R} R\Big[(\Delta_A - 1)\, r\mathrm{v}_i - \sum_{j=1}^{N} \varepsilon(\hat{\mathrm{w}}_j\, \hat{r}\, \hat{\mathrm{v}}_i)\, \mathrm{v}_j\Big]\,, \tag{5.15}$$

where L is regarded as an R-module via (5.14), *and* $R\,\mathcal{I} := R \otimes_A \mathcal{I}$.

Proof. If we identify $A \cong eBe \subset e\Pi e$, $\mathcal{I} \cong eBe_\infty \subset e\Pi e_\infty$ as in Lemma 5.4, the required isomorphism is induced by

$$R\,\mathcal{I} \xrightarrow{\pi_1} e\Pi e \otimes_{eBe} eBe_\infty \xrightarrow{\pi_2} e\Pi e_\infty \twoheadrightarrow e\Pi e_\infty \Big/ \sum_{u \in U} e\Pi e_\infty\,(u - \varepsilon(u))\,,$$

where π_1 is the product of (5.14) with $\mathcal{I}$ and π_2 is the multiplication map. $\square$

Now, the tensor algebra filtration on $R = T_A \mathbb{D}\mathrm{er}(A)$ induces the differential filtration on $\mathcal{D}$ via the canonical projection and module filtrations on L and $M \subseteq L$ via the isomorphism of Lemma 5.5. Writing $\bar{\mathcal{D}}, \bar{L}, \ldots$ for the associated graded objects relative to these filtrations, we have

$$\bar{M} \subseteq \bar{L} \cong \bar{R}\,\mathcal{I} / \bar{R}\,\bar{\Delta}_A \bar{R}\,\mathcal{I} \cong (\bar{R}/\bar{R}\,\bar{\Delta}_A\,\bar{R}) \otimes_A \mathcal{I} \cong \bar{\mathcal{D}} \otimes_A \mathcal{I} \cong \bar{\mathcal{D}}\,\mathcal{I}.$$

It follows that M is a rank 1 torsion-free module (as so is $\bar{M}$). Moreover, since $\dim \bar{L}/\bar{M} = \dim L/M < \infty$, by Theorem 3.1(a), $\gamma[M] = [\mathcal{I}]$. This completes Step 1.

5.3. The map ω is surjective

Given a rank 1 torsion-free $\mathcal{D}$-module M, we now construct a Π-module $\boldsymbol{L}$, together with a $\Pi^\lambda(B)$-module embedding $M \hookrightarrow \boldsymbol{L}$, such that $\boldsymbol{V} := \boldsymbol{L}/M$ has dimension $(n, 1)$ and $i^*[\boldsymbol{V}] \cong M$.

We begin with some preparations. We let $\widetilde{\mathcal{D}} := \bigoplus_{k=0}^{\infty} \mathcal{D}_k t^k$ denote the Rees algebra of the ring $\mathcal{D}$ with respect to its canonical filtration $\{\mathcal{D}_k\}$, and let $\mathsf{GrMod}(\widetilde{\mathcal{D}})$ be the category of graded $\widetilde{\mathcal{D}}$-modules. There is a natural homomorphism of graded rings $p : \widetilde{\mathcal{D}} \to \bar{\mathcal{D}}$, mapping $a\,t^k \in \widetilde{\mathcal{D}}_k$ to $a \,(\mathrm{mod}\, \mathcal{D}_{k-1}) \in \bar{\mathcal{D}}_k$. Using this homomorphism, we will regard graded $\bar{\mathcal{D}}$-modules as objects of $\mathsf{GrMod}(\widetilde{\mathcal{D}})$. Since $\operatorname{Ker}(p) = \langle\, t\,\rangle$, we may identify $\bar{\mathcal{D}} \cong \widetilde{\mathcal{D}}/\langle\, t\,\rangle$. This implies that $\widetilde{\mathcal{D}}$ is Noetherian, since so is $\bar{\mathcal{D}}$ (see [41], Proposition 3.5).

Next, following [2], we define $\mathsf{Tors}(\widetilde{\mathcal{D}})$ to be the full subcategory of $\mathsf{GrMod}(\widetilde{\mathcal{D}})$ consisting of torsion modules. By definition, $\widetilde{M} \in \mathsf{GrMod}(\widetilde{\mathcal{D}})$ is *torsion*, if for every $m \in \widetilde{M}$ there is $k_m \in \mathbb{N}$ such that $\widetilde{\mathcal{D}}_k\, m = 0$ for all $k \geq k_m$. By [2], Section 2, $\mathsf{Tors}(\widetilde{\mathcal{D}})$ is a localizing subcategory of $\mathsf{GrMod}(\widetilde{\mathcal{D}})$: i. e., the inclusion functor $\mathsf{Tors}(\widetilde{\mathcal{D}}) \hookrightarrow \mathsf{GrMod}(\widetilde{\mathcal{D}})$ has a right adjoint $\tau : \mathsf{GrMod}(\widetilde{\mathcal{D}}) \to \mathsf{Tors}(\widetilde{\mathcal{D}})$ which assigns to a graded module $\widetilde{M}$ its largest torsion submodule $\tau(\widetilde{M}) = \{m \in \widetilde{M} : \widetilde{\mathcal{D}}_k\, m = 0 \text{ for all } k \gg 0\}$. The functor τ is left exact, and we write $\tau_k := R^k\tau : \mathsf{GrMod}(\widetilde{\mathcal{D}}) \to \mathsf{Tors}(\widetilde{\mathcal{D}})$ for its derived functors.

We also introduce the quotient category $\mathsf{Qgr}(\widetilde{\mathcal{D}}) := \mathsf{GrMod}(\widetilde{\mathcal{D}})/\mathsf{Tors}(\widetilde{\mathcal{D}})$. This is an abelian category that comes equipped with two canonical functors: the (exact) localization functor $\pi : \mathsf{GrMod}(\widetilde{\mathcal{D}}) \to \mathsf{Qgr}(\widetilde{\mathcal{D}})$ and its right adjoint (and hence left exact) functor $\omega : \mathsf{Qgr}(\widetilde{\mathcal{D}}) \to \mathsf{GrMod}(\widetilde{\mathcal{D}})$. The relation between π, ω and τ is described by the following result which is part of standard torsion theory (see, for example, [2], Proposition 7.2).

Theorem 5.1. (1) *The adjunction map* $\eta_{\widetilde{M}} : \widetilde{M} \to \omega\,\pi(\widetilde{M})$ *fits into the exact sequence*

$$0 \to \tau(\widetilde{M}) \to \widetilde{M} \xrightarrow{\eta_{\widetilde{M}}} \omega\pi(\widetilde{M}) \to \tau_1(\widetilde{M}) \to 0 \tag{5.16}$$

which is functorial in $\widetilde{M} \in \mathsf{GrMod}(\widetilde{\mathcal{D}})$.

(2) *For* $k \geq 1$, *there are natural isomorphisms*

$$R^k\omega(\pi\widetilde{M}) \cong \tau_{k+1}(\widetilde{M}). \tag{5.17}$$

In particular, if $k \geq 1$, *the modules* $R^k\omega(\pi\widetilde{M})$ *are torsion.*

Now, given a graded module $\widetilde{M} = \bigoplus_{k\in\mathbb{Z}} \widetilde{M}_k$ and $n \in \mathbb{Z}$, we write $\widetilde{M}[n] := \bigoplus_{k\in\mathbb{Z}} \widetilde{M}_{k+n}$ and $\widetilde{M}_{\geq n} := \bigoplus_{k\geq n} \widetilde{M}_k$. Both are graded modules, $\widetilde{M}_{\geq n}$ being a submodule of $\widetilde{M}$. With this notation, we compute $R^k\omega(\pi\bar{\mathcal{D}})$, regarding $\bar{\mathcal{D}}$ as a $\widetilde{\mathcal{D}}$-module via the algebra map $p : \widetilde{\mathcal{D}} \to \bar{\mathcal{D}}$.

Lemma 5.6. (1) *The canonical map* $\eta_{\bar{\mathcal{D}}} : \bar{\mathcal{D}} \xrightarrow{\sim} \omega\pi(\bar{\mathcal{D}})_{\geq 0}$ *is an isomorphism.*

(2) $R^k\omega(\pi\bar{\mathcal{D}}) = 0$ *for* $k \geq 1$.

Proof. For graded $\bar{\mathcal{D}}$-modules $\bar{M}$ and $\bar{N}$, we define (*cf.* [2], Section 3)

$$\underline{\mathrm{Hom}}_{\bar{\mathcal{D}}}(\bar{M}, \bar{N}) := \bigoplus_{k\in Z} \mathrm{Hom}_{\mathrm{GrMod}(\bar{\mathcal{D}})}(\bar{M}, \bar{N}[k]),$$

and write $\underline{\mathrm{Ext}}^n_{\bar{\mathcal{D}}}(\bar{M}, \bar{N})$ for the corresponding Ext-groups. Combining [2], Theorem 8.3 and Proposition 7.2, we then identify

$$R^k\omega(\pi\bar{\mathcal{D}}) \cong \varinjlim \underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_{\geq n}, \bar{\mathcal{D}}), \quad \forall\, k \geq 0. \tag{5.18}$$

To compute the Ext-groups in (5.18) we use the long cohomology sequence

$$\underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}}) \to \underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_{\geq n}, \bar{\mathcal{D}}) \to \underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_{\geq n+1}, \bar{\mathcal{D}}) \to \underline{\mathrm{Ext}}^{k+1}_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}}) \tag{5.19}$$

arising from the short exact sequence $0 \to \bar{\mathcal{D}}_{\geq n+1} \to \bar{\mathcal{D}}_{\geq n} \to \bar{\mathcal{D}}_n[-n] \to 0$, and the following isomorphisms (for $n \geq 0$)

$$\underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}}) = 0 \quad \text{if} \quad k \neq 1, \tag{5.20}$$

and

$$\underline{\mathrm{Ext}}^1_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}})_m \cong \begin{cases} 0 & \text{if } m \neq -n-1 \\ \mathrm{Sym}^{-m}(\Omega^1 X) & \text{if } m = -n-1, \end{cases} \tag{5.21}$$

where Sym^q stands for the q-th symmetric power over A. It is easy to see that (5.19)–(5.21), together with (5.18), formally imply both statements of the lemma. (In addition, we have $\omega(\pi\bar{\mathcal{D}})_n \cong \mathrm{Sym}^{-n}(\Omega^1 X)$ for $n < 0$.)

To prove (5.20) and (5.21) we observe that $\underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n[-n], \bar{\mathcal{D}}) \cong \underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n, \bar{\mathcal{D}})[n]$, where $\bar{\mathcal{D}}_n$ is a graded $\bar{\mathcal{D}}$-module with a single component in degree 0. Such modules arise by restricting scalars via the algebra projection $\bar{\mathcal{D}} \to A$. So we can compute $\underline{\mathrm{Ext}}^k_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n, \bar{\mathcal{D}})$ using the spectral sequence

$$\mathrm{Ext}^p_A(\bar{\mathcal{D}}_n, \underline{\mathrm{Ext}}^q_{\bar{\mathcal{D}}}(A, \bar{\mathcal{D}})) \underset{p}{\Rightarrow} \underline{\mathrm{Ext}}^{p+q}_{\bar{\mathcal{D}}}(\bar{\mathcal{D}}_n, \bar{\mathcal{D}}). \tag{5.22}$$

To this end, we identify $\bar{\mathcal{D}}$ with the symmetric algebra $\mathrm{Sym}_A(\mathrm{Der}\,A)$ and use the canonical resolution

$$0 \to \bar{\mathcal{D}} \otimes_A \mathrm{Der}(A)[-1] \to \bar{\mathcal{D}} \to A \to 0. \tag{5.23}$$

It follows from (5.23) that $\underline{\mathrm{Ext}}^q_{\bar{\mathcal{D}}}(A, \bar{\mathcal{D}}) = 0$ for $q \neq 1$, so (5.22) collapses on the line $q = 1$, giving (after natural identifications) the isomorphisms (5.20) and (5.21). □

Lemma 5.7. *If* $\mathcal{I}$ *is a flat A-module, then*

$$R^k\omega\,\pi(\widetilde{M} \otimes_A \mathcal{I}) \cong R^k\omega\,(\pi\widetilde{M}) \otimes_A \mathcal{I}, \quad \forall\, k \geq 0,$$

for any graded $\widetilde{\mathcal{D}}$-*A-bimodule* $\widetilde{M}$.

Proof. By [2], Proposition 7.2(1), we have $R^k\omega\,\pi(\widetilde{M} \otimes_A \mathcal{I}) \cong \varinjlim \underline{\mathrm{Ext}}^k_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}_{\geq n}, \widetilde{M} \otimes_A \mathcal{I})$. Since $\varinjlim$ commutes with tensor products, it suffices to prove that

$$\underline{\mathrm{Ext}}^k_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}_{\geq n}, \widetilde{M} \otimes_A \mathcal{I}) \cong \underline{\mathrm{Ext}}^k_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}_{\geq n}, \widetilde{M}) \otimes_A \mathcal{I} \quad \text{for } n \gg 0. \tag{5.24}$$

Furthermore, by functoriality, it suffices to prove (5.24) only for $k = 0$, but in that case the result is well known (see [47], Lemma 3.83). □

Now, we turn to our problem. As in Section 3.2, we choose a good filtration $\{M_k\}$ on M so that $\bar{M} := \bigoplus_{k\in\mathbb{Z}} M_k/M_{k-1}$ is a torsion-free $\bar{\mathcal{D}}$-module. Then, by Theorem 3.1, there is an ideal $\mathcal{I} \subseteq A$ (unique up to isomorphism) and a graded embedding

$$\bar{f} : \bar{M} \hookrightarrow \bar{\mathcal{D}}\mathcal{I}, \tag{5.25}$$

such that $\dim \operatorname{Coker}(\bar{f}) < \infty$. The filtration $\{M_k\}$ is uniquely determined by M up to a shift of degree (*cf.* Lemma 5.12 below); we fix this shift by requiring $\bar{f}$ to be of degree 0. The dimension $n := \dim \operatorname{Coker}(\bar{f})$ is then an invariant of M, independent of the choice of filtration.

Since $\eta : \operatorname{Id} \to \omega\pi$ is a natural transformation, the map (5.25) fits into the commutative diagram

$$\begin{tikzcd} \bar{M} \arrow[r, "\bar{f}"] \arrow[d, "\eta_{\bar{M}}"'] & \bar{\mathcal{D}}\mathcal{I} \arrow[d, "\eta_{\bar{\mathcal{D}}\mathcal{I}}"] \\ \omega\pi(\bar{M}) \arrow[r, "\omega\pi(\bar{f})"] & \omega\pi(\bar{\mathcal{D}}\mathcal{I}) \end{tikzcd} \tag{5.26}$$

As $\operatorname{Ker}(\bar{f}) = 0$ and $\operatorname{Coker}(\bar{f}) \in \operatorname{Tors}(\widetilde{\mathcal{D}})$, $\pi(\bar{f})$ and, hence, $\omega\pi(\bar{f})$ are isomorphisms. On the other hand, by Lemma 5.7, $\eta_{\bar{\mathcal{D}}\mathcal{I}}$ can be factored as

$$\bar{\mathcal{D}}\mathcal{I} \cong \bar{\mathcal{D}} \otimes_A \mathcal{I} \xrightarrow{\eta_{\bar{\mathcal{D}}}\otimes 1} \omega\pi(\bar{\mathcal{D}}) \otimes_A \mathcal{I} \cong \omega\pi(\bar{\mathcal{D}}\mathcal{I})$$

and hence, by Lemma 5.6(1), $\eta_{\bar{\mathcal{D}}\mathcal{I}} : \bar{\mathcal{D}}\mathcal{I} \xrightarrow{\sim} \omega\pi(\bar{\mathcal{D}}\mathcal{I})_{\geq 0}$ is an isomorphism. Using these two isomorphisms, we identify

$$\omega\pi(\bar{M})_{\geq 0} \cong \bar{\mathcal{D}}\mathcal{I}. \tag{5.27}$$

It follows then from (5.16) and (5.26) that $\tau(\bar{M}) = 0$ and $\tau_1(\bar{M})_{\geq 0} \cong \operatorname{Coker}(\bar{f}) \cong \bar{\mathcal{D}}\mathcal{I}/\bar{M}$. Hence

$$\dim \tau_1(\bar{M})_{\geq 0} = n. \tag{5.28}$$

Next, we set $\widetilde{N} := \bigoplus_{k\in\mathbb{Z}} M/M_k$ and make $\widetilde{N}$ a graded $\widetilde{\mathcal{D}}$-module in the natural way, with $t \in \widetilde{\mathcal{D}}$ acting by the canonical projections $M/M_k \twoheadrightarrow M/M_{k+1}$.

Proposition 5.2. *The module $\widetilde{N}$ has the following properties:*
(1) $\tau(\widetilde{N}) = 0$,
(2) $\dim \tau_1(\widetilde{N})_{-1} = n$, *and* $\dim \tau_1(\widetilde{N})_{\geq -1} < \infty$,
(3) *The maps* $\omega\pi(\widetilde{N})_{k-1} \xrightarrow{t} \omega\pi(\widetilde{N})_k$ *are surjective for all* $k \geq 0$.

Proof. (1) Given $\widetilde{M} \in \operatorname{GrMod}(\widetilde{\mathcal{D}})$, we write $p^!(\widetilde{M})$ for the largest submodule of $\widetilde{M}$ annihilated by the action of t, i. e. $p^!(\widetilde{M}) = \operatorname{Ker}(\widetilde{M} \xrightarrow{t\cdot} \widetilde{M}[1])$. Then, if $\widetilde{M} \in \operatorname{Tors}(\widetilde{\mathcal{D}})$ and $\widetilde{M} \neq 0$, we have $p^!(\widetilde{M}) \neq 0$. So the assumption $\tau(\widetilde{N}) \neq 0$ implies that $p^!(\tau\widetilde{N}) \neq 0$. On the other hand, $p^!(\widetilde{N}) \cong \bar{M}[1]$ and $\tau(\bar{M}[1]) = \tau(\bar{M})[1] = 0$, so $\tau(p^!\widetilde{N}) = 0$. Since $p^!(\tau\widetilde{N}) = p^!(\widetilde{N}) \cap \tau(\widetilde{N}) = \tau(p^!\widetilde{N})$, we arrive at contradiction. It follows that $\tau(\widetilde{N}) = 0$.

(2) For all $k \in \mathbb{Z}$, we have the exact sequences $0 \to M_k/M_{k-1} \to M/M_{k-1} \to M/M_k \to 0$ defined by the filtration inclusions. Combining these together, we get the exact sequence of graded $\widetilde{\mathcal{D}}$-modules

$$0 \to \bar{M} \to \widetilde{N}[-1] \xrightarrow{t} \widetilde{N} \to 0. \tag{5.29}$$

Since $\tau(\widetilde{N}) = 0$, applying the torsion functor τ to (5.29) yields

$$0 \to \tau_1(\bar{M}) \to \tau_1(\widetilde{N}[-1]) \to \tau_1(\widetilde{N}) \to \tau_2(\bar{M}) \to \cdots \tag{5.30}$$

By Theorem 5.1(2) and Lemma 5.7, the last term of (5.30) can be identified as

$$\tau_2(\bar{M}) \cong R^1\omega(\pi\bar{M}) \cong R^1\omega[\pi(\bar{\mathcal{D}}\mathcal{I})] \cong R^1\omega(\pi\bar{\mathcal{D}}) \otimes_A \mathcal{I},$$

so $\tau_2(\bar{M}) = 0$ by Lemma 5.6(2). We get thus the exact sequence

$$0 \to \tau_1(\bar{M}) \to \tau_1(\widetilde{N}[-1]) \xrightarrow{t} \tau_1(\widetilde{N}) \to 0. \tag{5.31}$$

Now, (5.28) implies that $\tau_1(\bar{M})_{\geq 0}$ is bounded: i. e. there is an integer $d \geq 0$, such that $\tau_1(\bar{M})_d \neq 0$, while $\tau_1(\bar{M})_k = 0$ for all $k > d$. It follows then from (5.31) that t acts as a unit on $\tau_1(\widetilde{N})_{\geq d}$: in particular, we have $p^!(\tau_1(\widetilde{N})_{\geq d}) = 0$. But $\tau_1(\widetilde{N})_{\geq d}$ is a submodule of $\tau_1(\widetilde{N})$ which, by definition, is torsion. Hence, $p^!(\tau_1(\widetilde{N})_{\geq d}) = 0$ forces $\tau_1(\widetilde{N})_{\geq d} = 0$. Now, by induction, it follows from (5.31) that $\dim \tau_1(\widetilde{N})_{k-1} = \sum_{j=k}^{d} \dim \tau_1(\bar{M})_j$ for $k = 0, 1, \ldots, d$. In particular, by (5.28), $\dim \tau_1(\widetilde{N})_{-1} = \dim \tau_1(\bar{M})_{\geq 0} = n$, and $\dim \tau_1(\widetilde{N})_{\geq -1} = \sum_{k\geq 0} \dim \tau_1(\widetilde{N})_{k-1}$ is finite.

(3) Applying $\omega\pi$ to (5.29) gives rise to the exact sequence

$$0 \to \omega\pi(\bar{M}) \to \omega\pi(\widetilde{N}[-1]) \xrightarrow{t} \omega\pi(\widetilde{N}) \to R^1\omega(\pi\bar{M}) \to \cdots \tag{5.32}$$

Since $\pi(\bar{M}) \cong \pi(\bar{\mathcal{D}}\mathcal{I})$, we have $\omega\pi(\bar{M})_{\geq 0} \cong \omega\pi(\bar{\mathcal{D}}\mathcal{I})_{\geq 0} \cong \bar{\mathcal{D}}\mathcal{I}$, see (5.27), and $R^1\omega(\pi\bar{M}) \cong R^1\omega(\pi\bar{\mathcal{D}}\mathcal{I}) \cong R^1\omega(\pi\bar{\mathcal{D}}) \otimes_A \mathcal{I} = 0$, by Lemma 5.6(2). Hence, truncating (5.32) at negative degrees, we get the exact sequence

$$0 \to \bar{\mathcal{D}}\mathcal{I} \to \omega\pi(\widetilde{N}[-1])_{\geq 0} \xrightarrow{t} \omega\pi(\widetilde{N})_{\geq 0} \to 0. \tag{5.33}$$

The last statement of the proposition follows. $\square$

Next, we consider the functorial exact sequence (5.16), with $\widetilde{M} = \widetilde{N}$. By Proposition 5.2(1), the first term of this sequence is zero, so we have

$$0 \to \widetilde{N} \xrightarrow{\eta_{\widetilde{N}}} \omega\pi(\widetilde{N}) \to \tau_1(\widetilde{N}) \to 0, \tag{5.34}$$

Since the canonical filtration on M is positive, $\widetilde{N}_k = M$ for all $k < 0$. Thus, setting $L := \omega\pi(\widetilde{N})_{-1}$ and $V := \tau_1(\widetilde{N})_{-1}$, we get from (5.34) the exact sequence of A-modules

$$0 \to M \xrightarrow{\eta} L \to V \to 0. \tag{5.35}$$

Now, replacing A by its one-point extension $B = A[\mathcal{I}]$, we lift (5.35) to an exact sequence of B-modules, as follows. First, we regard M as a B-module by restricting scalars via the algebra homomorphism $i : B \to A$, see (2.13). Next, we set $\boldsymbol{L} := L \oplus \mathbb{C}$ and make $\boldsymbol{L}$ a B-module by defining its structure map $\varphi : \mathcal{I} \otimes \mathbb{C} \cong \mathcal{I} \to L$ to be the degree 0 component of the canonical embedding $\bar{\mathcal{D}}\mathcal{I} \hookrightarrow \omega\pi(\widetilde{N}[-1])_{\geq 0}$ in (5.33). Every A-module homomorphism $M \to L$ extends then to a unique B-module homomorphism $M \to \boldsymbol{L}$, since $\mathrm{Hom}_A(M, L) \cong \mathrm{Hom}_B(M, \boldsymbol{L})$ via $f \mapsto (f, 0)$. In particular, the map η in (5.35) extends to an embedding $\boldsymbol{\eta} : M \hookrightarrow \boldsymbol{L}$, and we write $\boldsymbol{V} := \boldsymbol{L}/M$ for the cokernel of $\boldsymbol{\eta}$. Clearly, $\boldsymbol{V} \cong V \oplus \mathbb{C}$ as a vector space, and $\underline{\dim}(\boldsymbol{V}) = (n, 1)$, by Proposition 5.2(2). Summing up, we have constructed an exact sequence of B-modules

$$0 \to M \xrightarrow{\boldsymbol{\eta}} \boldsymbol{L} \to \boldsymbol{V} \to 0, \tag{5.36}$$

with the quotient term being of dimension $(n, 1)$. Moreover, using Lemma 4.1, we may regard M as a $\Pi^{\boldsymbol{\lambda}}(B)$-module.

Proposition 5.3. *The B-module structure on $\boldsymbol{L}$ defined above admits a* unique *extension to $\Pi^{\boldsymbol{\lambda}}(B)$, making $\boldsymbol{\eta} : M \to \boldsymbol{L}$ a homomorphism of $\Pi^{\boldsymbol{\lambda}}(B)$-modules.*

We will give a homological proof of this proposition, using Theorem 2.2 of Section 2.1. As explained in (the proof of) Theorem 2.2, a $\Pi^{\boldsymbol{\lambda}}(B)$-module structure on a B-module $\boldsymbol{M}$ is determined by an element of $\mathrm{End}(\boldsymbol{M}) \otimes_{B^e} \Omega^1 B$, lying in the fibre of $\boldsymbol{\lambda}$ ($= \boldsymbol{\lambda} \cdot \mathrm{Id}$) under the evaluation map

$$\partial_{\boldsymbol{M}} : \mathrm{End}(\boldsymbol{M}) \otimes_{B^e} \Omega^1(B) \to \mathrm{End}(\boldsymbol{M}), \qquad f \otimes d \mapsto f\Delta_B(d). \tag{5.37}$$

In particular, the given $\Pi^{\boldsymbol{\lambda}}(B)$-module structure on M is determined by an element $\delta_M \in \mathrm{End}(M) \otimes_{B^e} \Omega^1(B)$, such that $\partial_M(\delta_M) = \mathrm{Id}_M$. The B-module embedding $\boldsymbol{\eta}$ induces an embedding of B-bimodules: $\mathrm{End}(M) \hookrightarrow \mathrm{Hom}(M, \boldsymbol{L})$, and hence the natural map

$$\mathrm{End}(M) \otimes_{B^e} \Omega^1(B) \hookrightarrow \mathrm{Hom}(M, \boldsymbol{L}) \otimes_{B^e} \Omega^1(B). \tag{5.38}$$

Since $\Omega^1(B)$ is a projective bimodule, this last map is also an embedding, and we identify $\mathrm{End}(M) \otimes_{B^e} \Omega^1(B)$ with its image in $\mathrm{Hom}(M, \boldsymbol{L}) \otimes_{B^e} \Omega^1(B)$ under (5.38).

Now, consider the commutative diagram

$$\begin{tikzcd} \mathrm{End}(\boldsymbol{L}) \otimes_{B^e} \Omega^1(B) \arrow[r, "\tilde{\boldsymbol{\eta}}_*"] \arrow[d, "\partial_{\boldsymbol{L}}"'] & \mathrm{Hom}(M, \boldsymbol{L}) \otimes_{B^e} \Omega^1(B) \arrow[d, "\partial_{M,\boldsymbol{L}}"'] \\ \mathrm{End}(\boldsymbol{L}) \arrow[r, "\boldsymbol{\eta}_*"] & \mathrm{Hom}(M, \boldsymbol{L}) \end{tikzcd} \tag{5.39}$$

where $\partial_{M,\boldsymbol{L}}$ is the evaluation map at Δ_B, $\boldsymbol{\eta}_*$ is the restriction (via $\boldsymbol{\eta}$), and $\tilde{\boldsymbol{\eta}}_* := \boldsymbol{\eta}_* \otimes \mathrm{Id}$. Note that $\boldsymbol{\eta}_*$ and $\tilde{\boldsymbol{\eta}}_*$ are both surjective. With above identification, we have

$$\boldsymbol{\eta}_*(\boldsymbol{\lambda}) = \partial_{M,\boldsymbol{L}}(\delta_M) = \boldsymbol{\eta}, \tag{5.40}$$

and our problem is to show that there is a unique element $\delta_{\boldsymbol{L}} \in \mathrm{End}(\boldsymbol{L}) \otimes_{B^e} \Omega^1(B)$, such that

$$\partial_{\boldsymbol{L}}(\delta_{\boldsymbol{L}}) = \boldsymbol{\lambda} \quad \text{and} \quad \tilde{\boldsymbol{\eta}}_*(\delta_{\boldsymbol{L}}) = \delta_M. \tag{5.41}$$

To solve this problem homologically, we interpret the top and the bottom rows of (5.39) as 2-complexes of vector spaces $X^\bullet$ and $Y^\bullet$, with nonzero terms in degrees 0 and 1 and differentials $\tilde{\boldsymbol{\eta}}_*$ and $\boldsymbol{\eta}_*$, respectively. The pair of maps $(\partial_{\boldsymbol{L}}, \partial_{M,\boldsymbol{L}})$ yields then a morphism of complexes $\partial^\bullet : X^\bullet \to Y^\bullet$ with mapping cone

$$C^\bullet(\partial) := \left[0 \to X^0 \xrightarrow{d^{-1}} X^1 \oplus Y^0 \xrightarrow{d^0} Y^1 \to 0 \right]. \tag{5.42}$$

By definition, the differentials in $C^\bullet(\partial)$ are given by $d^{-1} = (-\tilde{\boldsymbol{\eta}}_*,\ \partial_{\boldsymbol{L}})^T$ and $d^0 = (\partial_{M,\boldsymbol{L}},\ \boldsymbol{\eta}_*)$. So (5.40) can be interpreted by saying that $(-\delta_M, \boldsymbol{\lambda}) \in X^1 \oplus Y^0$ is a 0-cocycle in $C^\bullet(\partial)$. Then, the cohomology class

$$c(\boldsymbol{\lambda}, \delta_M) := [(-\delta_M, \boldsymbol{\lambda})] \tag{5.43}$$

represented by this cocycle, vanishes in $h^0(C^\bullet)$ if and only if there is $\delta_{\boldsymbol{L}} \in X^0$ such that $d^{-1}(\delta_{\boldsymbol{L}}) = (-\delta_M, \boldsymbol{\lambda})$, i. e. (5.41) holds. Clearly, if it exists, such $\delta_{\boldsymbol{L}}$ is unique if and only if d^{-1} is injective, i. e. if and only if $h^{-1}(C^\bullet) = 0$. Now, a simple calculation (as in the proof of Theorem 2.2) shows that

$$h^0(C^\bullet) \cong H_0(B,\ \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) \quad \text{and} \quad h^{-1}(C^\bullet) \cong H_1(B,\ \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})).$$

Proposition 5.3 thus boils down to proving Lemmas 5.8 and 5.9 below.

Lemma 5.8. *$H_1(B,\ \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) = 0$.*

Proof. Recall that $L := \omega\,\pi(\widetilde{N})_{-1}$. Now, in addition, we set $L_0 := \omega\,\pi(\widetilde{N})_0$ and make this a B-module by restricting scalars via $i : B \to A$. Then, the A-module homomorphism $t_L : L \to L_0$ induced by the action of t extends to a unique B-module homomorphism $\boldsymbol{L} \to L_0$ which we denote by $\boldsymbol{t}$. By Proposition 5.2(3), t_L is surjective, and hence so is $\boldsymbol{t}$. It is easy to see that $\mathrm{Ker}(\boldsymbol{t}) \cong Be_\infty$, so we have the exact sequence of B-modules

$$0 \to Be_\infty \xrightarrow{\iota} \boldsymbol{L} \xrightarrow{\boldsymbol{t}} L_0 \to 0. \tag{5.44}$$

Since Be_∞ is projective, tensoring (5.44) with $\boldsymbol{V}^* := \mathrm{Hom}(\boldsymbol{V},\ \mathbb{C})$ yields

$$0 \to \mathrm{Tor}_1^B(\boldsymbol{V}^*,\ \boldsymbol{L}) \to \mathrm{Tor}_1^B(\boldsymbol{V}^*,\ L_0) \to \boldsymbol{V}^* \otimes_B Be_\infty \to \boldsymbol{V}^* \otimes_B \boldsymbol{L} \to \boldsymbol{V}^* \otimes_B L_0 \to 0.$$

On the other hand, since $\boldsymbol{V}$ is finite-dimensional, for an arbitrary B-module $\boldsymbol{M}$, we have natural isomorphisms $H_n(B,\ \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{M})) \cong \mathrm{Tor}_n^B(\boldsymbol{V}^*,\ \boldsymbol{M})$. So the above exact sequence can be identified with

$$\begin{aligned} 0 \to H_1(B,\ \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) \to H_1(B,\ \mathrm{Hom}(\boldsymbol{V}, L_0)) \to \\ H_0(B,\ \mathrm{Hom}(\boldsymbol{V}, Be_\infty)) \to H_0(B,\ \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) \to H_0(B,\ \mathrm{Hom}(\boldsymbol{V}, L_0)) \to 0. \end{aligned} \tag{5.45}$$

To prove the lemma it thus suffices to show that

$$H_1(B,\ \mathrm{Hom}(\boldsymbol{V}, L_0)) \cong \mathrm{Tor}_1^B(\boldsymbol{V}^*,\ L_0) = 0. \tag{5.46}$$

Since L_0 is an A-module, we can compute this last Tor, using the spectral sequence

$$\mathrm{Tor}_p^A(\mathrm{Tor}_q^B(\boldsymbol{V}^*,\ A),\ L_0) \underset{p}{\Rightarrow} \mathrm{Tor}_{p+q}^B(\boldsymbol{V}^*,\ L_0) \tag{5.47}$$

associated to the algebra map $i : B \to A$. By Lemma 2.2(4), this map is flat, so (5.47) collapses at $q = 0$, giving an isomorphism

$$\mathrm{Tor}_1^B(\boldsymbol{V}^*,\ L_0) \cong \mathrm{Tor}_1^A(V^*,\ L_0). \tag{5.48}$$

Now, for each $k \geq 0$, we set $L_k := \omega\pi(\widetilde{N})_k$ and write F_k for the kernel of the map $L_0 \xrightarrow{t^k} L_k$ induced by the action of $t^k \in \widetilde{\mathcal{D}}$ on $\omega\pi(\widetilde{N})$. By Proposition 5.2(3), the maps t^k are surjective for all $k \geq 0$, and thus $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ is an A-module filtration on L_0. By Proposition 5.2(2), this filtration is exhaustive, so that $\varinjlim F_k \cong \bigcup_{k=0}^\infty F_k = L_0$, while, by (5.33), we have exact sequences

$$0 \to F_k \to F_{k+1} \to \overline{\mathcal{D}}_{k+1}\mathcal{I} \to 0, \quad \forall\, k \geq 0. \tag{5.49}$$

Since $\overline{\mathcal{D}}_{k+1}\mathcal{I}$ are projective A-modules for $k \geq 0$, we conclude from (5.49) that F_k are projective for $k \geq 1$. The direct limits of families of projective modules are flat, hence so is $L_0 = \varinjlim F_k$. We have $\mathrm{Tor}_1^A(V^*,\ L_0) = 0$, and (5.46) follows from (5.48). $\square$

Lemma 5.9. *$c(\boldsymbol{\lambda},\ \delta_M) = 0$ in $H_0(B,\ \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L}))$.*

Proof. By (5.45) and (5.46), we have the exact sequence

$$0 \to H_0(B, \mathrm{Hom}(\boldsymbol{V}, Be_\infty)) \xrightarrow{\iota_*} H_0(B, \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) \xrightarrow{\boldsymbol{t}_*} H_0(B, \mathrm{Hom}(\boldsymbol{V}, L_0)) \to 0, \tag{5.50}$$

where ι_* is induced by the inclusion $\iota : Be_\infty \hookrightarrow \boldsymbol{L}$ and $\boldsymbol{t}_*$ by the projection $\boldsymbol{t} : \boldsymbol{L} \twoheadrightarrow L_0$ in (5.44). We show first that $\boldsymbol{t}_*(c(\boldsymbol{\lambda},\ \delta_M)) = 0$. For this, we consider the image of the diagram (5.39) under the natural projection $\boldsymbol{t}$. Under this projection, the equations (5.40) become

$$\boldsymbol{\eta}_*(\boldsymbol{t}) = \partial_{M,L_0}(\tilde{\boldsymbol{t}}_*(\delta_M)) = \boldsymbol{t} \circ \boldsymbol{\eta}, \tag{5.51}$$

where $\tilde{\boldsymbol{t}}_* : \mathrm{Hom}(M, \boldsymbol{L}) \otimes_{B^e} \Omega^1(B) \to \mathrm{Hom}(M, L_0) \otimes_{B^e} \Omega^1(B)$ is defined by $f \otimes \omega \mapsto (\boldsymbol{t} \circ f) \otimes \omega$. Now, $\boldsymbol{t}$ induces a morphism of mapping cones (5.42) associated to (5.39) and its projection which, in turn, induces a map $\boldsymbol{t}_*$ on cohomology.

The class $\boldsymbol{t}_*(c(\boldsymbol{\lambda},\ \delta_M)) \in H_0(B, \mathrm{Hom}(\boldsymbol{V}, L_0))$ can thus be viewed as an obstruction for the existence of an element $\delta_{\boldsymbol{L},L_0} \in \mathrm{Hom}(\boldsymbol{L}, L_0) \otimes_{B^e} \Omega^1(B)$ satisfying

$$\partial_{\boldsymbol{L},L_0}(\delta_{\boldsymbol{L},L_0}) = \boldsymbol{t} \quad \text{and} \quad \tilde{\boldsymbol{\eta}}_*(\delta_{\boldsymbol{L},L_0}) = \tilde{\boldsymbol{t}}_*(\delta_M). \tag{5.52}$$

We will show that $\boldsymbol{t}_*(c(\boldsymbol{\lambda},\ \delta_M)) = 0$ by constructing such an element explicitly.

By the universal property of tensor algebras, the filtered algebra homomorphism $R \twoheadrightarrow \mathcal{D}$ lifts to a *graded* algebra homomorphism $R \twoheadrightarrow \widetilde{\mathcal{D}}$, so we may regard graded $\widetilde{\mathcal{D}}$-modules as graded R-modules. The action of $\widetilde{\mathcal{D}}$ on $\omega\pi(\widetilde{N})$ yields an A-bimodule map $\mathbb{D}\mathrm{er}(A) \to \mathrm{Hom}(L,\ L_0)$, taking $\Delta_A \mapsto t_L$ which in composition with $\mathbb{D}\mathrm{er}(B) \to \mathbb{D}\mathrm{er}(A)$ gives a B-bimodule homomorphism $\mathbb{D}\mathrm{er}(B) \to \mathrm{Hom}(L,\ L_0)$, $\Delta_B \mapsto t_L$, and hence an element

$$\delta_{L,L_0} \in \mathrm{Hom}_{B^e}(\mathbb{D}\mathrm{er}(B),\ \mathrm{Hom}(L,\ L_0)) \cong \mathrm{Hom}(L,\ L_0) \otimes_{B^e} \Omega^1(B). \tag{5.53}$$

Now, let $\boldsymbol{\alpha} : L = e\boldsymbol{L} \hookrightarrow \boldsymbol{L}$ be the natural inclusion, with $\boldsymbol{L}/L = e_\infty \boldsymbol{L} \cong \mathbb{C}$. Viewing L as a B-module via i makes it a B-module map. Dualizing $\boldsymbol{\alpha}$ by L_0 and tensoring with $\Omega^1(B)$, we get then the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\boldsymbol{L}, L_0) \otimes_{B^e} \Omega^1(B) & \xrightarrow{\tilde{\boldsymbol{\alpha}}_*} & \mathrm{Hom}(L, L_0) \otimes_{B^e} \Omega^1(B) \\ \downarrow{\scriptstyle \partial_{\boldsymbol{L},L_0}} & & \downarrow{\scriptstyle \partial_{L,L_0}} \\ \mathrm{Hom}(\boldsymbol{L}, L_0) & \xrightarrow{\boldsymbol{\alpha}_*} & \mathrm{Hom}(L, L_0) \end{array} \tag{5.54}$$

with $\partial_{L,L_0}(\delta_{L,L_0}) = \boldsymbol{\alpha}_*(\boldsymbol{t}) = t_L$. Since $e \in B$ acts as identity on L_0 and as zero on $\boldsymbol{L}/L$, we have $H_0(B,\ \mathrm{Hom}(\boldsymbol{L}/L, L_0)) \cong (\boldsymbol{L}/L)^* \otimes_B L_0 = 0$. Hence, there is $\delta_{\boldsymbol{L},L_0} \in \mathrm{Hom}(\boldsymbol{L}, L_0) \otimes_{B^e} \Omega^1(B)$, such that $\partial_{\boldsymbol{L},L_0}(\delta_{\boldsymbol{L},L_0}) = \boldsymbol{t}$ and $\tilde{\boldsymbol{\alpha}}_*(\delta_{\boldsymbol{L},L_0}) = \delta_{L,L_0}$. A direct calculation using $\boldsymbol{\eta} = \boldsymbol{\alpha} \circ \eta$ shows that this element satisfies also (5.52).

The existence of $\delta_{\boldsymbol{L},L_0}$ implies that $\boldsymbol{t}_*(c(\boldsymbol{\lambda}, \delta_M)) = 0$. Returning to (5.50), we see then that $c(\boldsymbol{\lambda}, \delta_M) = \boldsymbol{\iota}_*(\tilde{c})$ for some $\tilde{c} \in H_0(B, \mathrm{Hom}(\boldsymbol{V}, Be_\infty))$. Now, to show that $\tilde{c} = 0$ we consider the trace map $\mathrm{Tr} : \mathrm{Hom}(\boldsymbol{V}, Be_\infty) \to \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L}) \to \mathrm{End}(\boldsymbol{V}) \to \mathbb{C}, f \mapsto \mathrm{tr}_{\boldsymbol{V}}[\boldsymbol{\pi} \circ \boldsymbol{\iota} \circ f]$, where $\boldsymbol{\iota}$ is defined in (5.44), $\boldsymbol{\pi}$ is the canonical projection in (5.36). Since $\boldsymbol{\iota}$ and $\boldsymbol{\pi}$ are homomorphisms, this induces a linear map

$$\mathrm{Tr}_* :\ H_0(B, \mathrm{Hom}(\boldsymbol{V}, Be_\infty)) \xrightarrow{\boldsymbol{\iota}_*} H_0(B, \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{L})) \xrightarrow{\boldsymbol{\pi}_*} H_0(B, \mathrm{End}(\boldsymbol{V})) \xrightarrow{\mathrm{tr}_*} \mathbb{C}.$$

We claim that Tr_* is an isomorphism. Indeed, it is easy to see that $\mathrm{Tr}_* \neq 0$, while

$$H_0(B, \mathrm{Hom}(\boldsymbol{V}, Be_\infty)) \cong \boldsymbol{V}^* \otimes_B Be_\infty \cong \boldsymbol{V}^* e_\infty \cong (e_\infty \boldsymbol{V})^* \cong \mathbb{C}.$$

Now, since $\boldsymbol{\pi} \circ \boldsymbol{\eta} = 0$, we have $\boldsymbol{\pi}_*(c(\boldsymbol{\lambda}, \delta_M)) = [\boldsymbol{\lambda} \cdot \mathrm{Id}_{\boldsymbol{V}}]$, and hence

$$\mathrm{Tr}_*(\tilde{c}) := \mathrm{tr}_{\boldsymbol{V}}[\boldsymbol{\pi}_*\, \boldsymbol{\iota}_*(\tilde{c})] = \mathrm{tr}_{\boldsymbol{V}}[\boldsymbol{\pi}_*(c)] = \mathrm{tr}_{\boldsymbol{V}}[\boldsymbol{\lambda} \cdot \mathrm{Id}_{\boldsymbol{V}}] = 0.$$

It follows that $\tilde{c} = 0$ and $c(\boldsymbol{\lambda},\ \delta_M) = 0$, finishing the proof of the lemma and Proposition 5.3. □

Now, by Proposition 5.3, the given B-module structure on $\boldsymbol{V} = \boldsymbol{L}/M$ extends to a unique $\Pi^{\boldsymbol{\lambda}}(B)$-module structure, making (5.36) an exact sequence of $\Pi^{\boldsymbol{\lambda}}(B)$-modules. Since $e_\infty \boldsymbol{V} \cong e_\infty \boldsymbol{L}$, the natural map $\Pi e_\infty \otimes_U e_\infty \boldsymbol{V} \cong \Pi e_\infty \otimes_U e_\infty \boldsymbol{L} \to \boldsymbol{L}$ is an isomorphism which in combination with projection $\boldsymbol{\pi} : \boldsymbol{L} \to \boldsymbol{V}$ becomes $\boldsymbol{\mu}_{\boldsymbol{V}} : \Pi e_\infty \otimes_U e_\infty \boldsymbol{V} \to \boldsymbol{V}$. It follows that $\mathrm{Ker}(\boldsymbol{\pi}) \cong \mathrm{Ker}(\boldsymbol{\mu}_{\boldsymbol{V}})$, and thus $M \cong i^*(\boldsymbol{V})$. This completes Step 2.

5.4. The map ω is injective and Γ-equivariant

For Π-modules $\boldsymbol{V}$ and $\boldsymbol{V}'$ of dimension $\boldsymbol{n} = (n, 1)$, we will show that

$$i^*(\boldsymbol{V}) \cong i^*(\boldsymbol{V}') \quad \Longleftrightarrow \quad \boldsymbol{V}' \cong \boldsymbol{V}^{\sigma_\omega} \quad \text{for some } \omega = u^{-1}du \in \Omega^1 X, \tag{5.55}$$

where $\boldsymbol{V}^\sigma$ denotes the Π-module $\boldsymbol{V}$ twisted by an automorphism $\sigma \in \mathrm{Aut}_S\, \Pi^{\boldsymbol{\lambda}}(B)$.

We begin by describing the action (4.17) in terms of generators of $\Pi^{\boldsymbol{\lambda}}(B)$ (see Proposition 5.1).

Lemma 5.10. *The homomorphism $\sigma :\ \Omega^1 X \to \mathrm{Aut}_S\, \Pi$ is given by*

$$\sigma_\omega(\hat{a}) = \hat{a}, \quad \sigma_\omega(\hat{\mathrm{v}}_i) = \hat{\mathrm{v}}_i, \qquad \sigma_\omega(\hat{\mathrm{w}}_i) = \hat{\mathrm{w}}_i, \qquad \sigma_\omega(\hat{d}) = \hat{d} + \widehat{\omega(d)}, \tag{5.56}$$

where $\omega \in \Omega^1 X$ acts on $d \in \mathbb{D}\mathrm{er}(A)$ via the natural identification

$$\Omega^1 X = (\Omega^1 A)_\natural \cong \mathrm{Hom}_{A^e}((\Omega^1 A)^\star,\ A) \cong \mathrm{Hom}_{A^e}(\mathrm{Der}(A,\ A^{\otimes 2}),\ A).$$

Proof. By Lemma 5.1, we can define (4.17) in terms of relative differentials

$$\sigma :\ \Omega^1 X \ \xrightarrow{\alpha}\ (\Omega^1_S B)_\natural \ \xrightarrow{\tilde{\sigma}}\ \mathrm{Aut}_B[T_B(\Omega^1_S B)^\star] \to \mathrm{Aut}_S\, \Pi^{\boldsymbol{\lambda}}(B), \tag{5.57}$$

where α is now an isomorphism. In fact, with identification (5.8), the elements of $(\Omega^1_S B)_\natural = \Omega^1_S B/[B,\ \Omega^1_S B]$ can be represented by matrices $\hat{\omega} = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}$ with $\omega \in (\Omega^1 A)_\natural = \Omega^1 X$, and α is given explicitly by $\omega \mapsto \hat{\omega} \bmod [B,\ \Omega^1_S B]$. It follows then that σ_ω acts on $\Pi^{\boldsymbol{\lambda}}(B)$ as in (5.56). □

We may also describe the algebra map $i : \Pi^{\lambda}(B) \to \mathcal{D}$ in terms of generators of $\Pi^{\lambda}(B)$:

$$i(\hat{a}) = \bar{a}, \qquad i(\hat{d}) = \bar{d}, \qquad i(\hat{\mathrm{v}}_i) = i(\hat{\mathrm{w}}_i) = 0, \tag{5.58}$$

where $\bar{a}$ and $\bar{d}$ denote the classes of $a \in A$ and $d \in \mathbb{D}\mathrm{er}(A)$ in $T_A\mathbb{D}\mathrm{er}(A)$ modulo the ideal $\langle \Delta_A - 1\rangle$. Comparing now (5.56) and (5.58), we get

Lemma 5.11. *The group homomorphism $\bar{\sigma} : \Omega^1X \xrightarrow{\sigma} \mathrm{Aut}_S\,\Pi \to \mathrm{Aut}_{\mathbb{C}}\,\mathcal{D}$ induced by σ is given by*

$$\bar{\sigma}_{\omega}(a) = a, \qquad \bar{\sigma}_{\omega}(\partial) = \partial + \omega(\partial), \tag{5.59}$$

where $a \in A$, $\partial \in \mathrm{Der}(A)$ and $\omega \in \Omega^1X$.

In particular, if $\omega = u^{-1}du$ for some $u \in \Lambda$, then $\bar{\sigma}_{\omega}(a) = a = u\,a\,u^{-1}$ (since A is commutative), and $\bar{\sigma}_{\omega}(\partial) = u\,\partial\,u^{-1}$. Thus, the induced action of $\Lambda \subset \Omega^1X$ on $\mathcal{D}$ is given by *inner* automorphisms. In contrast, Λ does not act by inner automorphisms on the whole of $\Pi^{\lambda}(B)$.

Now, by functoriality, $\boldsymbol{V}' \cong \boldsymbol{V}^{\sigma_\omega}$ implies $\boldsymbol{L}' \cong \boldsymbol{L}^{\sigma_\omega}$ and $i^*(\boldsymbol{V}') \cong i^*(\boldsymbol{V})^{\sigma_\omega}$ for any $\omega \in \Omega^1X$. So the map $\mathcal{C}_n(X, \mathcal{I}) \to \mathcal{I}(\mathcal{D})$ induced by i^* is equivariant under the action of Ω^1X. On the other hand, $i^*(\boldsymbol{V})$ is a $\mathcal{D}$-module on which the twisting by ω acts via (5.59), i. e. $i^*(\boldsymbol{V})^{\sigma_\omega} = i^*(\boldsymbol{V})^{\bar{\sigma}_\omega}$. Since the inner automorphisms induce trivial auto-equivalences, we have $i^*(\boldsymbol{V})^{\sigma_\omega} \cong i^*(\boldsymbol{V})$ for $\omega = u^{-1}du$. This proves the implication '$\Leftarrow$' in (5.55) and, in combination with Step 1, yields a Γ-equivariant map $\omega_n : \overline{\mathcal{C}}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}]$.

It remains to show that ω_n is injective. For this, we will use the following result which is a version of [16], Lemma 10.1, and [46], Lemma 3.2. (In particular, the proof given in the last reference extends trivially to our situation.)

Lemma 5.12. *Let M be a (nonzero) ideal of $\mathcal{D}$ equipped with two good filtrations $\{M_k\}$ and $\{M'_k\}$, such that the associated graded modules $\bar{M}$ and $\bar{M}'$ are both torsion-free. Then, there is $k_0 \in \mathbb{Z}$, such that $M_k = M'_{k-k_0}$ for all $k \in \mathbb{Z}$.*

Given two Π-modules $\boldsymbol{V}$ and $\boldsymbol{V}'$ of dimension $\boldsymbol{n}$, we set $\boldsymbol{L} := \Pi e_\infty \otimes_U e_\infty \boldsymbol{V}$, $L := e\boldsymbol{L}$, $M := i^*(\boldsymbol{V})$, and similarly for $\boldsymbol{V}'$. In addition, we denote by $\eta : M \hookrightarrow L$ and $\boldsymbol{\eta} : M \hookrightarrow \boldsymbol{L}$ the natural inclusions (and similarly for M').

Proposition 5.4. *If $M \cong M'$ as $\mathcal{D}$-modules, then $\boldsymbol{L} \cong \boldsymbol{L}'$ as B-modules.*

Proof. First, we show that every $\mathcal{D}$-module isomorphism $f : M \to M'$ lifts to an A-module isomorphism $f_L : L \to L'$. For this, we identify L as in Lemma 5.5, filter it by $\{F_kL\}$ as in Section 5.2, and set $\widetilde{L} := \bigoplus_{k\in\mathbb{Z}} L/F_kL$. By (5.15), we have $\Delta_A \cdot x \equiv x\,(\mathrm{mod}\,F_0L)$ for all $x \in L$, so $\Delta_A\,[x]_k = [x]_{k+1} = t\,[x]_k$ for $k \geq -1$. Since $R[t]/\langle \Delta_A - t\rangle \cong \widetilde{\mathcal{D}}$, we may regard $\widetilde{L}_{\geq -1}$ as a graded $\widetilde{\mathcal{D}}$-module.

Next, we equip M with the induced filtration $M_k := M \cap F_kL$ via the inclusion $\eta : M \hookrightarrow L$, and put $\widetilde{N} := \bigoplus_{k\in\mathbb{Z}} M/M_k$. The map η naturally extends to $\tilde{\eta} : \widetilde{N} \hookrightarrow \widetilde{L}$, and $\widetilde{N}$ becomes a graded $\widetilde{\mathcal{D}}$-module via the induced action of $R[t]$ on $\widetilde{L}$. It follows from Lemma 5.5 that $\bar{M} := \bigoplus_{k\in\mathbb{Z}} M_k/M_{k+1}$ is a torsion-free $\bar{\mathcal{D}}$-module, and hence $\tau(\widetilde{N}) = 0$ by Proposition 5.2(1). Let $\eta_{\widetilde{N}} : \widetilde{N} \hookrightarrow \omega\pi(\widetilde{N})$, see (5.16). Since Coker $\tilde{\eta}$ is finite-dimensional in degree ≥ -1, the map $\eta_{\widetilde{N}}$ extends to an embedding: $\widetilde{L}_{\geq -1} \hookrightarrow \omega\pi(\widetilde{N})_{\geq -1}$. By induction in grading, using Proposition 5.2(2) and (5.33), it is easy to show that this embedding is an isomorphism.

Now, replacing L by L', we repeat the above construction. The $\mathcal{D}$-module M' comes then equipped with two filtrations: one is induced from L' via $\eta' : M' \hookrightarrow L'$, and the other is transferred from M via $f : M \xrightarrow{\sim} M'$. Both filtrations satisfy the assumptions of Lemma 5.12 and, hence, coincide up to a shift in degree. Since $\bar{M}'$ and $f(\bar{M})$ have finite codimension in $\bar{L}'$, this last shift must be 0 so $M'_k = f(M_k)$ for all $k \in \mathbb{Z}$. The map f extends then to an isomorphism $\tilde{f} : \widetilde{N} \to \widetilde{N}'$ and further, by functoriality, to $\omega\pi(\tilde{f}) : \omega\pi(\widetilde{N}) \to \omega\pi(\widetilde{N}')$. As a result, we get $\widetilde{L}_{\geq -1} \cong \omega\pi(\widetilde{N})_{\geq -1} \xrightarrow{\sim} \omega\pi(\widetilde{N}')_{\geq -1} \cong \widetilde{L}'_{\geq -1}$ which in degree (-1) yields the required extension $f_L : L \to L'$.

Now, with our identifications of L and L', the B-modules $\boldsymbol{L}$ and $\boldsymbol{L}'$ are determined (up to isomorphism) by the triples $(L, \mathbb{C}, \varphi)$ and $(L', \mathbb{C}, \varphi')$, where $\varphi : \mathcal{I} \hookrightarrow L$ and $\varphi' : \mathcal{I} \hookrightarrow L'$ are the canonical embeddings with images F_0L and F_0L' respectively. Since F_0L is the kernel of $\widetilde{L}_{-1} \overset{t}{\twoheadrightarrow} \widetilde{L}_0$, the map f_L restricts to F_0L, giving an isomorphism $f_L|_0 : F_0L \to F_0L'$. Letting $u := (\varphi')^{-1} \circ (f_L|_0) \circ \varphi \in \mathrm{Aut}_A(\mathcal{I})$ and identifying $\mathrm{Aut}_A(\mathcal{I}) = \mathrm{End}_A(\mathcal{I})^{\times} \cong A^{\times}$ via the action map, we have $u\,\varphi' = \varphi' u = f_L\,\varphi$. Hence

$$\boldsymbol{g} := (u^{-1}f_L,\ \mathrm{Id}) : L \oplus \mathbb{C} \to L' \oplus \mathbb{C} \tag{5.60}$$

makes the diagram (2.11) commutative and thus defines an isomorphism of B-modules $\boldsymbol{L} \xrightarrow{\sim} \boldsymbol{L}'$. $\square$

Now, keeping the notation of Proposition 5.4, consider two Π-modules $\boldsymbol{V}$ and $\boldsymbol{V}'$ of dimension $\boldsymbol{n}$, with $M \cong M'$. Fix an isomorphism $f : M \to M'$ and define $\boldsymbol{g}$ as in (5.60). Taking $\omega = u^{-1}du \in \Omega^1X$ and twisting $\boldsymbol{\eta}$ by $\sigma = \sigma_\omega \in \mathrm{Aut}_S\,\Pi$, consider the diagram

$$\begin{array}{ccc} \boldsymbol{L}^{\sigma} & \xrightarrow{\ \boldsymbol{g}\ } & \boldsymbol{L}' \\ \boldsymbol{\eta}\uparrow & & \uparrow\boldsymbol{\eta}' \\ M^{\sigma} & \xrightarrow{\ fu^{-1}\ } & M' \end{array} \tag{5.61}$$

From the construction of f and $\boldsymbol{g}$, it follows that this diagram is commutative, with all arrows being Π-*module* homomorphisms and horizontal ones being isomorphisms. Thus, identifying $M^\sigma \cong M'$ and $\boldsymbol{L}^\sigma \cong \boldsymbol{L}'$ in (5.61), we get two (*a priori* different) Π-module structures on $\boldsymbol{L}'$. Both of these are extensions of the given Π-module structure on M'. Hence, by Proposition 5.3, they must coincide. It follows that $\boldsymbol{g} : \boldsymbol{L}^\sigma \to \boldsymbol{L}'$ is an isomorphism of Π-modules which, by commutativity of (5.61), induces an isomorphism $\boldsymbol{V}^\sigma \cong \boldsymbol{V}'$. This completes Step 3.

5.5. *The equivariance of* ω *under the action of* $\mathrm{Pic}(\mathcal{D})$

As in Section 4, we will assume that $X \neq \mathbb{A}^1$. By [20], Proposition 1.4, the automorphism group of $\mathcal{D}$ is then isomorphic to the product $\mathrm{Aut}(X) \ltimes \Omega^1 X$:

$$\mathrm{Aut}(X) \ltimes \Omega^1 X \xrightarrow{\sim} \mathrm{Aut}(\mathcal{D}), \quad (\nu,\ \omega) \mapsto \bar{\nu}\,\bar{\sigma}_\omega, \tag{5.62}$$

where $\bar{\nu} \in \mathrm{Aut}(\mathcal{D}) : D \mapsto \nu\, D\, \nu^{-1}$, and $\bar{\sigma}_\omega$ is defined by (5.59). Now, for a line bundle $\mathcal{F}$ on X, $\mathrm{End}_{\mathcal{D}}(\mathcal{F}\,\mathcal{D})$ is canonically isomorphic to the ring of twisted differential operators on X with coefficients in $\mathcal{F}$. As X is affine, this last ring is isomorphic to $\mathcal{D}$, so the set of all algebra isomorphisms: $\mathcal{D} \to \mathrm{End}_{\mathcal{D}}(\mathcal{F}\,\mathcal{D})$ is non-empty and equals $\psi_0\,\mathrm{Aut}(\mathcal{D})$, where ψ_0 is a fixed isomorphism. By [20], Theorem 1.8, the isomorphism ψ_0 can be chosen in such a way that $\psi_0\,|_A = \mathrm{Id}$: specifically, fixing dual bases $\{\alpha_i\} \subset \mathcal{F}$, $\{\beta_i\} \subset \mathcal{F}^\vee$, and identifying $\mathrm{End}_{\mathcal{D}}(\mathcal{F}\,\mathcal{D}) = \mathcal{F}\,\mathcal{D}\,\mathcal{F}^\vee$ as in Section 4.2, we define $\psi_0 : \mathcal{D} \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}}(\mathcal{F}\,\mathcal{D})$ by

$$\psi_0(a) = a, \qquad \psi_0(\partial) = \sum_i \alpha_i\,\partial\,\beta_i, \quad a \in A,\ \partial \in \mathrm{Der}(A). \tag{5.63}$$

With (5.62) and (5.63), every isomorphism $\psi : \mathcal{D} \to \mathrm{End}_{\mathcal{D}}(\mathcal{F}\,\mathcal{D})$ can then be decomposed as

$$\psi = \psi_0\,\bar{\nu}\,\bar{\sigma}_\omega, \tag{5.64}$$

where $\nu \in \mathrm{Aut}(X)$ and $\omega \in \Omega^1 X$ are uniquely determined by ψ.

Proof of Proposition 4.2. Given a line bundle $\mathcal{I}$ and an invertible bimodule $\mathcal{P} = (\mathcal{D}\mathcal{L})_\varphi$, with $\varphi : \mathcal{D} \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}}(\mathcal{D}\mathcal{L})$, we set $\tau := \varphi|_A$, $\mathcal{J} := \mathcal{L}\tau(\mathcal{I})$, $\mathcal{F} := \mathcal{L}^\tau = \tau^{-1}(\mathcal{L})$, and $\psi = \varphi^{-1} : \mathcal{D} \to \mathrm{End}_{\mathcal{D}}(\mathcal{F}\,\mathcal{D})$, as in Section 4.2. To construct an isomorphism $\tilde{\psi}$, satisfying Lemma 4.2, we decompose ψ as in (5.64), and extend each factor through i. Since ψ_0 and $\bar{\sigma}_\omega$ act on A as identity, we have $\nu = \psi|_A = \tau^{-1}$, so $\bar{\nu} = \bar{\tau}^{-1}$ in (5.64). Thus we set

$$\widetilde{\psi}\ :\ \Pi^\lambda(A[\mathcal{J}]) \xrightarrow{\sigma_\omega} \Pi^\lambda(A[\mathcal{J}]) \xrightarrow{\tilde{\tau}^{-1}} \Pi^\lambda(A[\mathcal{J}^\tau]) \xrightarrow{\widetilde{\psi}_0} \mathrm{End}_{\Pi^\lambda(B)}(\boldsymbol{P}),$$

where σ_ω is defined in Section 4.2 (see (4.17), with B replaced by $A[\mathcal{J}]$) and $\tilde{\tau}^{-1}$ is induced by $A[\mathcal{J}] \to A[\mathcal{J}^\tau]$. The relation $i\,\sigma_\omega = \bar{\sigma}_\omega\,i$ is then immediate, by Lemma 5.11.

It remains to define $\widetilde{\psi}_0$. To this end, we use identification (4.5). Since $\mathcal{J}^\tau = \mathcal{F}\,\mathcal{I}$, we have then $A[\mathcal{J}^\tau] \cong \tilde{\mathcal{F}} \otimes_{\tilde{A}} B \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee \hookrightarrow \tilde{\mathcal{F}} \otimes_{\tilde{A}} \Pi^\lambda(B) \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee$, which we take as a definition of $\widetilde{\psi}_0$ on $A[\mathcal{J}^\tau]$. This induces the identity on A, as required. Next, we construct a bimodule isomorphism:

$$\mathrm{Der}_S(A[\mathcal{F}\,\mathcal{I}],\ A[\mathcal{F}\,\mathcal{I}]^{\otimes 2}) \to \tilde{\mathcal{F}} \otimes_{\tilde{A}} \mathbb{D}\mathrm{er}_S(B) \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee, \tag{5.65}$$

using the dual bases for $\mathcal{F}$ and $\mathcal{I}$. By Lemma 5.2, we first identify the domain of (5.65) with

$$\begin{pmatrix} \mathbb{D}\mathrm{er}(A) & \mathrm{Der}(A,\ \mathcal{F}\,\mathcal{I} \otimes A) \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} \mathcal{F}\,\mathcal{I} \otimes (\mathcal{F}\,\mathcal{I})^\vee & \mathcal{F}\,\mathcal{I} \otimes A \\ (\mathcal{F}\,\mathcal{I})^\vee & A \end{pmatrix} \tag{5.66}$$

and the codomain with

$$\begin{pmatrix} \mathcal{F} \otimes \mathbb{D}\mathrm{er}(A) \otimes \mathcal{F}^\vee & \mathrm{Der}(A,\ \mathcal{I} \otimes \mathcal{F}) \\ 0 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} \mathcal{F}\,\mathcal{I} \otimes (\mathcal{F}\,\mathcal{I})^\vee & \mathcal{F}\,\mathcal{I} \otimes A \\ (\mathcal{F}\,\mathcal{I})^\vee & A \end{pmatrix}.$$

The first summand of (5.66) is generated by the elements $\hat{d} \in e\,\mathbb{D}\mathrm{er}(A)\,e$ (see Proposition 5.1): so we define the map (5.65) on this first summand by

$$\hat{d} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \sum_i \alpha_i \otimes d \otimes \beta_i & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.67}$$

while letting it be the identity on the second. This yields an isomorphism of bimodules and induces the required algebra map $\widetilde{\psi}_0$. The commutativity $i\,\widetilde{\psi}_0 = \psi_0\,i$ is verified by an easy calculation, using (5.58).

To finish the proof of Proposition 4.2 it remains to show the uniqueness of $\tilde{\psi}$. For this, arguing as in Proposition 5.3, it suffices to show that $H_1(A[\mathcal{J}],\ \mathrm{Ker}\, i^\otimes) = 0$, where $i^\otimes := 1 \otimes i \otimes 1$, see (4.7). Since $A[\mathcal{J}] \cong \tilde{\mathcal{F}} \otimes_{\tilde{A}} B \otimes_{\tilde{A}} \tilde{\mathcal{F}}^\vee$, see (4.6), we may identify $H_1(A[\mathcal{J}],\ \mathrm{Ker}\, i^\otimes) \cong H_1(B,\ \mathrm{Ker}\, i)$. On the other hand, by Lemma 5.4, $\mathrm{Ker}\, i \cong \Pi^\lambda(B)e_\infty \otimes_U e_\infty \Pi^\lambda(B)$ which is easily seen to be a flat B-bimodule. Thus $H_1(B,\ \mathrm{Ker}\, i) = 0$, as required. $\square$

Proof of Proposition 4.3. (1) We will keep the notation of Proposition 4.2. For $\mathcal{P} = \mathcal{D}_{\bar{\sigma}_\omega}$, we have then $\mathcal{L} \cong A$, $\varphi = \bar{\sigma}_\omega$, $\tau = \mathrm{Id}_A$ and $\psi = \bar{\sigma}_\omega^{-1}$. Now, since $\mathcal{F} = \mathcal{L}^\tau \cong A$, we may choose $\psi_0 = \mathrm{Id}_{\mathcal{D}}$. Then $\widetilde{\psi} = \sigma_\omega^{-1}$, and the bimodule $\boldsymbol{P}$ is isomorphic to $\Pi^\lambda(B)$ with left multiplication twisted by σ_ω^{-1}. Hence, for $\mathcal{P} = \mathcal{D}_{\bar{\sigma}_\omega}$, the isomorphism (4.9) is given by $[\boldsymbol{V}] \mapsto [\boldsymbol{V}^{\sigma_\omega^{-1}}]$ which agrees with our definition of σ_ω^*, see (4.18).

(2) For $\mathcal{P} = (\mathcal{D}\mathcal{L})_\varphi$, the map $f_{\mathcal{P}} : \mathcal{C}_n(X, \mathcal{I}) \to \mathcal{C}_n(X, \mathcal{J})$ is equivariant under Λ in the sense that

$$f_{\mathcal{P}} \circ \sigma_\omega^* = \sigma_{\omega_\tau}^* \circ f_{\mathcal{P}}, \quad \forall\, u \in \Lambda, \tag{5.68}$$

where $\omega = \mathtt{dlog}(u)$ and $\omega_\tau = \mathtt{dlog}[\tau(u)]$. Indeed, $f_{\mathcal{P}} \circ \sigma_\omega^*$ is induced by tensoring Π-modules with the bimodule ${}_\psi\boldsymbol{P}_{\sigma_\omega} = {}_\psi\boldsymbol{P} \otimes_\Pi \Pi_{\sigma_\omega}$ on which $\Pi^\lambda(A[\mathcal{J}])$ acts on the left via $\widetilde{\psi}$. Since $\tilde{\tau}\,\sigma_\omega = \sigma_{\omega_\tau}\,\tilde{\tau}$, we have ${}_\psi\boldsymbol{P}_{\sigma_\omega} \cong {}_{\sigma_\omega^{-1}\psi}\boldsymbol{P} \cong {}_{\psi\sigma_{\omega_\tau}^{-1}}\boldsymbol{P} \cong (\Pi'_{\sigma_{\omega_\tau}}) \otimes_{\Pi'} ({}_\psi\boldsymbol{P})$, where $\Pi' := \Pi^\lambda(A[\mathcal{J}])$. This implies (5.68). Now, it follows from (5.68) that $f_{\mathcal{P}}$ induces a well-defined map $\bar{f}_{\mathcal{P}}$ on the quotient varieties. The map $\bar{f}_{\mathcal{P}}$ depends only on the class $[\mathcal{P}] \in \mathrm{Pic}(\mathcal{D})$, since $[\mathcal{P}]$ determines φ (and $\psi = \varphi^{-1}$) up to an *inner* automorphism of $\mathcal{D}$. By Proposition 4.2, this means that $\widetilde{\psi}$ (and hence, $f_{\mathcal{P}}$) are determined by $[\mathcal{P}]$ up to an automorphism $\sigma_\omega \in \mathrm{Aut}_S[\Pi']$ with $\omega = \mathtt{dlog}(u)$, $u \in \Lambda$. Since such automorphisms act trivially on $\overline{\mathcal{C}}_n(X, \mathcal{I})$, the map $\bar{f}_{\mathcal{P}}$ is uniquely determined by $[\mathcal{P}] \in \mathrm{Pic}(\mathcal{D})$. $\square$

Finally, we prove the last part of Theorem 4.2.

Proof of Theorem 4.2(c). Let $\boldsymbol{V}$ be a $\Pi^\lambda(B)$-module representing a point of $\mathcal{C}_n(X, \mathcal{I})$. The class $\omega_n[\boldsymbol{V}] \in \gamma^{-1}[\mathcal{I}]$ can then be represented by an ideal M fitting into the exact sequence

$$0 \to M \to \boldsymbol{L} \to \boldsymbol{V} \to 0, \tag{5.69}$$

where $\boldsymbol{L} = \Pi e_\infty \otimes_U e_\infty \boldsymbol{V}$. Now, given an invertible bimodule $\mathcal{P} = (\mathcal{D}\mathcal{L})_\varphi$, we write $\Pi' = \Pi^\lambda(A[\mathcal{J}])$, $U' = e_\infty \Pi' e_\infty$ and observe that $\boldsymbol{P} \otimes_\Pi (\Pi e_\infty \otimes_U e_\infty \Pi) \otimes_\Pi \boldsymbol{P}^* \cong \Pi' e_\infty \otimes_{U'} e_\infty \Pi'$, where $\boldsymbol{P}$ is the progenerator from Π to Π' determined by $\mathcal{P}$. On the other hand, we have

$${}_\psi\boldsymbol{P} \otimes_\Pi \mathcal{D} \cong {}_\psi(\tilde{\mathcal{F}} \otimes_{\tilde{A}} \Pi \otimes_\Pi \mathcal{D}) \cong {}_\psi(\tilde{\mathcal{F}} \otimes_{\mathcal{D}} \mathcal{D}) \cong {}_\psi(\mathcal{F}\mathcal{D}) \cong (\mathcal{D}\mathcal{L})_\varphi = \mathcal{P}.$$

Tensoring now (5.69) with $\boldsymbol{P}$ shows that the Π'-modules $\boldsymbol{V}' := \boldsymbol{P} \otimes_\Pi \boldsymbol{V}$ and $M' := \mathcal{P} \otimes_{\mathcal{D}} M$ fit into the exact sequence $0 \to M' \to \boldsymbol{L}' \to \boldsymbol{V}' \to 0$, with $\boldsymbol{L}' = \Pi' e_\infty \otimes_{U'} e_\infty \boldsymbol{V}'$. This means that $[M'] \in \gamma^{-1}[\mathcal{J}]$ corresponds under ω_n to $[\boldsymbol{V}'] \in \mathcal{C}_n(X, \mathcal{J})$, verifying the commutativity of (4.20) and finishing the proof of Theorem 4.2. $\square$

6. Explicit construction of ideals. Examples

6.1. Distinguished representatives

Given a rank 1 torsion-free $\mathcal{D}$-module M, we choose an embedding $e : M \hookrightarrow Q$, where $Q = \mathrm{Frac}(\mathcal{D})$. Such an embedding is unique up to automorphism of Q. We will fix this automorphism at a later stage of our calculation. Now, regarding M and Q as modules over $R = T_A\mathbb{D}\mathrm{er}(A)$, we may try to extend e to L through $\eta : M \hookrightarrow L$. It is easy to see, however, that such an extension does not exist in $\mathtt{Mod}(R)$. On the other hand, we have

Lemma 6.1. *There is a unique A-linear map $e_L : L \to Q$ extending e in* $\mathtt{Mod}(A)$.

Proof. Let $\eta_* : \mathrm{Hom}_A(L, Q) \to \mathrm{Hom}_A(M, Q)$ be the restriction map. We have $\mathrm{Ker}(\eta_*) \cong \mathrm{Hom}_A(V, Q) = 0$, since V is a torsion A-module, while Q is torsion-free. On the other hand, $\mathrm{Coker}(\eta_*)$ is isomorphic to a submodule of $\mathrm{Ext}^1_A(V, Q)$, while $\mathrm{Ext}^1_A(V, Q) = 0$, since Q is an injective A-module. It follows that η_* is an isomorphism. $\square$

Our aim is to compute e_L explicitly, in terms of representation V. First, we consider the map

$$\mathrm{ad} : \mathrm{Hom}_A(L, Q) \to \mathrm{Der}_A(R, \mathrm{Hom}(L, Q)), \tag{6.1}$$

taking $f : L \to Q$ to the inner derivation $\mathrm{ad}_f(r)(x) := rf(x) - f(rx)$, where $r \in R$ and $x \in L$. Since $\mathrm{Ker}(\mathrm{ad}) \cong \mathrm{Hom}_R(L, Q) = 0$, the map (6.1) is injective, and every $f \in \mathrm{Hom}_A(L, Q)$ is uniquely determined by ad_f. In addition, if f restricts to an R-linear map $M \to Q$, then $\eta_*(\mathrm{ad}_f) = 0$ in $\mathrm{Der}_A(R, \mathrm{Hom}(M, Q))$, and ad_f is determined by a (unique) derivation in $\mathrm{Der}_A(R, \mathrm{Hom}(V, Q))$. Thus e_L is uniquely determined by $\delta_V \in \mathrm{Der}_A(R, \mathrm{Hom}(V, Q))$ satisfying

$$e_L(rx) - re_L(x) = \delta_V(r)[\pi(x)], \quad \forall\, r \in R, \forall\, x \in L, \tag{6.2}$$

where $\pi : L \to V$. Furthermore, by the Leibniz rule, the restriction map

$$\mathrm{Der}_A(R, \mathrm{Hom}(V, Q)) \xrightarrow{\sim} \mathrm{Hom}_{A^e}(\mathbb{D}\mathrm{er}(A), \mathrm{Hom}(V, Q))$$

is an isomorphism: we thus need to compute δ_V on $\mathbb{D}\mathrm{er}(A)$ only.

Let $\mathbb{C}(X \times X)^{\mathrm{reg}}$ be the subring of rational functions on $X \times X$, regular outside the diagonal of $X \times X$. Geometrically, we can think of $\Omega^1(A) \subset A^{\otimes 2}$ as the ideal of the diagonal in $X \times X$, and $\Omega^1(A)^\star := \mathrm{Hom}_{A^{\otimes 2}}(\Omega^1 A, A^{\otimes 2})$ as the subspace of functions in $\mathbb{C}(X \times X)^{\mathrm{reg}}$ with (at most) simple poles along the diagonal; the canonical pairing between $\Omega^1 A$ and $\Omega^1(A)^\star$ is the given by multiplication in $\mathcal{O}(X \times X)$. Translating this into algebraic language, we have

Lemma 6.2. *Let $\flat$ be the involution on $\mathbb{C}(X \times X)^{\mathrm{reg}}$ induced by interchanging the factors in $X \times X$.*
(1) *The assignment $d \mapsto [d(a)/(a \otimes 1 - 1 \otimes a)\,]^{\flat}$ defines an injective bimodule homomorphism $\nu : \mathbb{D}\mathrm{er}(A) \to \mathbb{C}(X \times X)^{\mathrm{reg}}$.*
(2) *If $a \in A$, $d \in \mathbb{D}\mathrm{er}(A)$ and $d(a) = \sum_j f_j \otimes g_j$, then $[d,\ a] = \sum_j g_j\, \Delta_A f_j$.*

Now, to compute $\delta_V(d) \in \mathrm{Hom}(V, Q)$ we identify $\mathrm{Hom}(V, Q) \cong Q \otimes V^*$. There is a natural action of $R^{\mathrm{e}} := R \otimes R^{\circ}$ on this space: $R^{\mathrm{e}} \to Q \otimes \mathrm{End}(V^*)$, which is the tensor product of the dual representation $\varrho^* : R^{\circ} \to \mathrm{End}(V^*)$ with composition of natural maps $R \twoheadrightarrow \Pi^1(A) \cong \mathcal{D} \hookrightarrow Q$. Abusing notation, we will write $a \otimes b^*$ for the image of $a \otimes b^{\circ} \in R^{\mathrm{e}}$ in $Q \otimes \mathrm{End}(V^*)$. Restricting to $A^{\otimes 2} \subset R^{\mathrm{e}}$, we now get a ring homomorphism $A^{\otimes 2} \to Q \otimes \mathrm{End}(V^*)$. Since $\dim(V) < \infty$, this homomorphism takes the elements $a \otimes 1 - 1 \otimes a$, with $a \in A \setminus \mathbb{C}$, to units in $Q \otimes \mathrm{End}(V^*)$ and hence extends canonically to

$$\mathbb{C}(X \times X)^{\mathrm{reg}} \to Q \otimes \mathrm{End}(V^*).$$

Combining this last homomorphism with the embedding of Lemma 6.2, we define a bimodule map

$$\nu_V : \mathbb{D}\mathrm{er}(A) \to Q \otimes \mathrm{End}(V^*), \quad \Delta_A \mapsto 1 \otimes \mathrm{Id}_{V^*}. \tag{6.3}$$

We can now compute δ_V in terms of ν_V. To this end, we choose dual bases $\{\mathrm{v}_i\}$ and $\{\mathrm{w}_i\}$ for $\mathcal{I}$ and $\mathcal{I}^{\vee}$; by Proposition 5.1, this gives generators $\hat{a}$, $\hat{d}$, $\hat{\mathrm{v}}_i$ and $\hat{\mathrm{w}}_i$ for Π. Identifying $L_\infty \cong V_\infty \cong \mathbb{C}$, we think of $\hat{\mathrm{v}}_i$ and $\hat{\mathrm{w}}_i$ acting on $\boldsymbol{L}$ as linear maps $\mathrm{v}_i : \mathbb{C} \to L$ and $\mathrm{w}_i : L \to \mathbb{C}$, i. e. as elements of L and L^*. Similarly, when acting on $\boldsymbol{V}$, $\hat{\mathrm{v}}_i$ and $\hat{\mathrm{w}}_i$ give rise to vectors $\bar{\mathrm{v}}_i \in V$ and covectors $\bar{\mathrm{w}}_i \in V^*$. Note that $\bar{\mathrm{v}}_i = \pi \mathrm{v}_i$ and $\bar{\mathrm{w}}_i \pi = \mathrm{w}_i$, where $\pi : L \twoheadrightarrow V$. Further, we fix $\mathcal{I} \hookrightarrow A$ and identify L as in Lemma 5.5. Then we twist $e : M \hookrightarrow Q$ by an automorphism of Q in such a way that $e_L(\mathrm{v}) = \mathrm{v}$ for all $\mathrm{v} \in \mathcal{I} \subset A \subset Q$. This is possible, since $e_L : L \to Q$ is an A-linear extension of e, by Lemma 6.1. With this notation, we have

Proposition 6.1. *The derivation $\delta_V : \mathbb{D}\mathrm{er}(A) \to Q \otimes V^*$ is given by $\delta_V(d) = \sum_i \nu_V(d)[\mathrm{v}_i\, \bar{\mathrm{w}}_i]$.*

Proof. First, using the fact that Δ_A acts as $1 + \sum_i \mathrm{v}_i \mathrm{w}_i$ on L and as identity on Q, it is easy to compute $\delta_V(\Delta_A) = \sum_i \mathrm{v}_i \bar{\mathrm{w}}_i$. Now, if $r = [d,\ a] \in \mathbb{D}\mathrm{er}(A)$, then $\delta_V(r) = [\delta_V(d), a]$, since $\delta_V(a) = 0$. On the other hand, by Lemma 6.2(2), we have $[d, a] = \sum_j g_j\, \Delta_A f_j$, so $\delta_V(r) = \sum_j g_j\, \delta_V(\Delta_A) f_j$. Thus, $[\delta_V(d), a] = \sum_j g_j\, \delta_V(\Delta_A) f_j$, or, if we think of $\delta_V(d)$ as an element of $Q \otimes V^*$, then $(1 \otimes a^* - a \otimes 1)\, \delta_V(d) = \left(\sum_j g_j \otimes f_j^*\right)\, \delta_V(\Delta_A)$. Lemma 6.2(1) shows now that $\delta_V(d) = \nu_V(d)\, [\delta_V(\Delta_A)]$. $\square$

Now, we can state the main result of this section. For $\mathrm{v} \in \mathcal{I}$ and $d \in \mathbb{D}\mathrm{er}(A)$, we define

$$\kappa(d, \mathrm{v}) := \mathrm{v} - (1 \otimes d^* - d \otimes 1)^{-1}\, \delta_V(d)\, [1 \otimes \bar{\mathrm{v}}]\ \in Q, \tag{6.4}$$

where $\bar{\mathrm{v}} = \pi(\mathrm{v}) \in V$ and $(1 \otimes d^* - d \otimes 1)^{-1} \in Q \otimes \mathrm{End}(V^*)$.

Theorem 6.1. *Let $\boldsymbol{V}$ be a $\Pi^{\boldsymbol{\lambda}}(B)$-module of dimension $\boldsymbol{n} = (n, 1)$ representing a point in $\mathcal{C}_n(X,\ \mathcal{I})$. Then the class $\omega[\boldsymbol{V}] \in \mathcal{I}(\mathcal{D})$ can be represented by the (fractional) ideal M generated by the elements $\det_{V^*}(1 \otimes a^* - a \otimes 1)\, \mathrm{v}$ and $\det_{V^*}(1 \otimes d^* - d \otimes 1)\, \kappa(d, \mathrm{v})$, where $a \in A$, $d \in \mathbb{D}\mathrm{er}(A)$ and $\mathrm{v} \in \mathcal{I}$.*

Theorem 6.1 needs some explanations.

1. Formally, by (6.4), $\kappa(d, \mathrm{v})$ is well defined only when $1 \otimes d^* - d \otimes 1$ is invertible in $Q \otimes \mathrm{End}(V^*)$. It is easy to see, however, that the product $\det_{V^*}(1 \otimes d^* - d \otimes 1)\, \kappa(d, \mathrm{v}) \in M$ makes sense for all $d \in \mathbb{D}\mathrm{er}(A)$ (*cf.* [11], Remark 2, p. 83).

2. For generators of M it suffices to take the above determinants with a, d and v from some (finite) sets generating A, $\mathbb{D}\mathrm{er}(A)$ and the ideal $\mathcal{I}$.

Proof. By (5.13), the class $\omega(\boldsymbol{V})$ can be represented by $\widetilde{M} = \mathrm{Ker}[\pi : L \to V]$. Our goal is to show that the two kinds of determinants given in the proposition generate $M := e_L(\widetilde{M})$. To simplify the notation, we denote the elements of $\mathcal{I}$ (resp., $\mathcal{I}^{\vee}$) and the corresponding elements of V (resp., V^*) by the same letter. Using the Leibniz rule, for any $r \in R$ and $m \geq 1$, we have

$$\delta_V(r^m) = \left(\sum_{s=0}^{m-1} r^s \otimes (r^*)^{m-s-1}\right) \delta_V(r) = \frac{1 \otimes (r^*)^m - r^m \otimes 1}{1 \otimes r^* - r \otimes 1}\, \delta_V(r)\,, \tag{6.5}$$

provided $1 \otimes r^* - r \otimes 1 \in Q \otimes \mathrm{End}(V^*)$ is invertible. Now, consider the characteristic polynomial $p(t) = \chi_r(t) := \det_{\rho}(r - t\, \mathrm{Id}_V)$ of $r \in R$ in the representation V. It is clear that, for any $x \in L$, $p(r)x$ lies in the kernel of $\pi : L \twoheadrightarrow V$, thus $p(r)x \in \widetilde{M}$. To compute its image under e_L, we write $e_L(p(r)x) = p(r)e_L(x) + \delta_V(p(r))[1 \otimes \bar{x}]$, where $\bar{x} = \pi(x)$. Using (6.5) and the fact that $p(t) = \chi_r(t)$ annihilates $r^* \in \mathrm{End}(V^*)$, we get $\delta_V(p(r)) = -(p(r) \otimes 1)(1 \otimes r^* - r \otimes 1)^{-1}\delta_V(r)$. As a result, for $x = \mathrm{v} \in \mathcal{I}$,

$$e_L(\chi_r(r)\mathrm{v}) = \chi_r(r)\left(\mathrm{v} - (1 \otimes r^* - r \otimes 1)^{-1}\delta_V(r)[1 \otimes \bar{\mathrm{v}}]\right) \in M. \tag{6.6}$$

Choosing different $r \in R$, we obtain in this way various elements of M. In particular, for $r = a \in A$, we have $\delta_V(a) = 0$, so (6.6) produces the elements of the first kind $\chi_a(a)\mathrm{v} \in M$. On the other hand, taking $r = d \in \mathbb{D}\mathrm{er}(A)$ results in $\chi_d(d)\kappa(d, \mathrm{v})$ which are the elements of the second kind in M.

Finally, a simple filtration argument shows that the elements $\chi_a(a)\mathrm{v}$ and $\chi_d(d)\mathrm{v}$, with a,d and v running over some generating sets of A, $\mathbb{D}\mathrm{er}(A)$ and $\mathcal{I}$, generate a submodule $\widetilde{N} \subset \widetilde{M}$ of finite codimension in L. Hence $\widetilde{N} = \widetilde{M}$, and the images of these elements generate thus $M = e_L(\widetilde{M})$. □

6.2. Examples

6.2.1. The affine line

Let $X = \mathbb{A}^1$. Choosing a global coordinate on X, we identify $A = \mathcal{O}(X) \cong \mathbb{C}[x]$. In this case, $\mathbb{D}\mathrm{er}(A)$ is a free bimodule of rank 1; as a generator of $\mathbb{D}\mathrm{er}(A)$, we may take the derivation y defined by $y(x) = 1 \otimes 1$. It is easy to check that $\Delta_A = yx - xy$ in $\mathbb{D}\mathrm{er}(A)$. The algebra $R = T_A\mathbb{D}\mathrm{er}(A)$ is isomorphic to the free algebra $\mathbb{C}\langle x,\ y\rangle$, and $\Pi^1(A) \cong \mathbb{C}\langle x, y\rangle/\langle xy - yx + 1\rangle$ is the Weyl algebra $A_1(\mathbb{C})$. The map ν of Lemma 6.2 is given by

$$\nu(y) = (1 \otimes x - x \otimes 1)^{-1}, \qquad \nu(\Delta) = 1.$$

All line bundles on X are trivial, so we only need to consider $B = A[\mathcal{I}]$ with $\mathcal{I} = A$. The n-th Calogero–Moser variety $\mathcal{C}_n := \mathcal{C}_n(X, A)$ can be described as the space of equivalence classes of matrices

$$\{(\bar{X}, \bar{Y}, \bar{\mathrm{v}}, \bar{\mathrm{w}}) : \bar{X} \in \mathrm{End}(\mathbb{C}^n),\ \bar{Y} \in \mathrm{End}(\mathbb{C}^n),\ \bar{\mathrm{v}} \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n),\ \bar{\mathrm{w}} \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})\},$$

satisfying the relation $\bar{Y}\bar{X} - \bar{X}\bar{Y} = \mathrm{Id}_n + \bar{\mathrm{v}}\bar{\mathrm{w}}$, modulo the natural action of $\mathrm{GL}_n(\mathbb{C})$:

$$(\bar{X},\ \bar{Y},\ \bar{\mathrm{v}},\ \bar{\mathrm{w}}) \ \mapsto\ (g\bar{X}g^{-1},\ g\bar{Y}g^{-1},\ g\bar{\mathrm{v}},\ \bar{\mathrm{w}}g^{-1}), \quad g \in \mathrm{GL}_n(\mathbb{C}).$$

If we choose $\mathrm{v} = 1$ as a generator of $\mathcal{I} = A$, then the ideal M of $\mathcal{D} \cong \Pi^1(A)$ corresponding to a point $(\bar{X}, \bar{Y}, \bar{\mathrm{v}}, \bar{\mathrm{w}})$ is given by

$$M = \mathcal{D} \cdot \det(\bar{X} - x\,\mathrm{Id}_n) \ +\ \mathcal{D} \cdot \det(\bar{Y} - y\,\mathrm{Id}_n)\,\kappa,$$

where $\kappa = 1 - \bar{\mathrm{v}}^t(\bar{Y}^t - y\,\mathrm{Id}_n)^{-1}(\bar{X}^t - x\,\mathrm{Id}_n)^{-1}\bar{\mathrm{w}}^t$. This agrees with the description of ideals of $A_1(\mathbb{C})$ given in [11].

6.2.2. The complex torus

Let $X = \mathbb{C}^*$. We identify $A = \mathcal{O}(X)$ with $\mathbb{C}[x, x^{-1}]$, the ring of Laurent polynomials. As in the affine line case, $\mathbb{D}\mathrm{er}(A)$ is freely generated by the derivation y defined by $y(x) = 1 \otimes 1$. The algebra R is isomorphic to the free product $\mathbb{C}\langle x^{\pm 1}, y\rangle := \mathbb{C}[x, x^{-1}] \star \mathbb{C}[y]$, and $\Delta_A = yx - xy$ in R. The matrix description of the Calogero–Moser spaces $\mathcal{C}_n$ and the formulae for the corresponding fractional ideals of $\mathcal{D} \cong \Pi^1(A) = \mathbb{C}\langle x^{\pm 1}, y\rangle/\langle xy - yx + 1\rangle$ are the same as above, except for the fact that x and $\bar{X}$ are now invertible. A new feature is that A has nontrivial units x^r, $r \in \mathbb{Z}$. The corresponding group Λ can be identified with $\mathbb{Z}$ and its action on $\mathcal{C}_n$ is given by

$$r.(\bar{X},\ \bar{Y},\ \bar{\mathrm{v}},\ \bar{\mathrm{w}}) = (\bar{X},\ \bar{Y} + r\bar{X}^{-1},\ \bar{\mathrm{v}},\ \bar{\mathrm{w}}), \quad r \in \mathbb{Z}.$$

Thus, by Theorem 4.2, the classes of ideals of $\mathcal{D} \cong \Pi^1(A)$ are parametrized by the points of the quotient variety $\overline{\mathcal{C}}_n = \mathcal{C}_n/\mathbb{Z}$. It is worth mentioning that one may choose a different generator for the bimodule $\mathbb{D}\mathrm{er}(A)$: for example, $z = yx$, instead of y. Then $\Delta_A = z - xzx^{-1}$, which gives an alternative matrix description of $\mathcal{C}_n$ and the corresponding ideals.

6.2.3. A general plane curve

Let X be a smooth curve in $\mathbb{C}^2$ defined by the equation $F(x, y) = 0$, with $F(x, y) := \sum_{r,s} a_{rs}x^r y^s \in \mathbb{C}[x, y]$. In this case, the algebra $A \cong \mathbb{C}[x, y]/\langle F(x, y)\rangle$ is generated by x and y and the module $\mathrm{Der}(A)$ is (freely) generated by the derivation ∂ defined by $\partial(x) = F'_y(x, y)$, $\partial(y) = -F'_x(x, y)$. The bimodule $\mathbb{D}\mathrm{er}(A)$ is generated by the derivation $\Delta = \Delta_A$ and the element z defined by

$$z(x) = \sum_{r,s} a_{rs}\frac{x^r y^s \otimes 1 - x^r \otimes y^s}{y \otimes 1 - 1 \otimes y}, \qquad z(y) = -\sum_{r,s} a_{rs}\frac{x^r \otimes y^s - 1 \otimes x^r y^s}{x \otimes 1 - 1 \otimes x}.$$

These generators satisfy the following commutation relations

$$[z,\ x] = \sum_{r,s} a_{rs}\sum_{k=0}^{s-1} y^{s-k-1}\Delta y^k x^r, \qquad [z,\ y] = -\sum_{r,s} a_{rs}\sum_{l=0}^{r-1} y^s x^{r-l-1}\Delta x^l. \tag{6.7}$$

By Proposition 5.1, the algebra $\Pi^{\lambda}(B)$ is then generated by the elements $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{\mathrm{v}}_i$, $\hat{\mathrm{w}}_i$ and $\hat{\Delta}$, subject to the relations (5.10) and (6.7). The assignment $x \mapsto x$, $y \mapsto y$, $z \mapsto \partial$, $\Delta \mapsto 1$ extends to an isomorphism between $\Pi^1(A)$ and the ring $\mathcal{D}$ of differential operators on X. The bimodule map ν of Lemma 6.2 is given by

$$\nu(z) = -\frac{\sum_{r,s} a_{rs}y^s \otimes x^r}{(1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)}, \qquad \nu(\Delta) = 1. \tag{6.8}$$

Now, let us describe generic points of the varieties $\mathcal{C}_n(X, \mathcal{I})$; for simplicity, we consider only the case when $\mathcal{I}$ is trivial. Choose n distinct points $p_i = (x_i, y_i) \in X$, $i = 1, \ldots, n$, and define

$$(\bar{X}, \bar{Y}, \bar{Z}, \bar{\mathrm{v}}, \bar{\mathrm{w}}) \in \mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n) \times \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \times \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}) \tag{6.9}$$

by the following formulae

$$\bar{X} = \mathrm{diag}(x_1, \ldots, x_n), \qquad \bar{Y} = \mathrm{diag}(y_1, \ldots, y_n), \qquad \bar{\mathrm{v}}^t = -\bar{\mathrm{w}} = (1, \ldots, 1), \tag{6.10}$$

$$\bar{Z}_{ii} = \alpha_i \quad \text{and} \quad \bar{Z}_{ij} = \frac{F(x_j, y_i)}{(x_i - x_j)(y_i - y_j)} \quad (\text{for } i \neq j),$$

where $\alpha_1, \ldots, \alpha_n$ are arbitrary scalars. Then, a straightforward calculation, using the relations (6.7), shows that the assignment

$$\hat{x} \mapsto \bar{X}, \qquad \hat{y} \mapsto \bar{Y}, \qquad \hat{z} \mapsto \bar{Z}, \qquad \hat{\mathrm{v}} \mapsto \bar{\mathrm{v}}, \qquad \hat{\mathrm{w}} \mapsto \bar{\mathrm{w}}, \qquad \hat{\Delta} \mapsto \mathrm{Id}_n + \bar{\mathrm{v}}\,\bar{\mathrm{w}}$$

extends to a representation of $\Pi^{\boldsymbol{\lambda}}(B)$, with $B = A[A]$ and $\boldsymbol{\lambda} = (1, -n)$, on the vector space $\boldsymbol{V} = \mathbb{C}^n \oplus \mathbb{C}$.

Remark. The matrix $\bar{Z}$ defined above is a generalization of the classical *Moser matrix* in the theory of integrable systems (see [36]).

To illustrate Theorem 6.1 we now describe the fractional ideal representing the class $\omega[\boldsymbol{V}]$ for an arbitrary $[\boldsymbol{V}] \in \mathcal{C}_n(X, \mathcal{I})$. We consider first the case when $\mathcal{I}$ is trivial. In that case, we identify $\mathcal{I} = \mathcal{I}^\vee = A$ and choose $\mathrm{v} = \mathrm{w} = 1$ as the generators of $\mathcal{I}$ and $\mathcal{I}^\vee$. A representation $\boldsymbol{V} = \mathbb{C}^n \oplus \mathbb{C}$ may then be described by the matrices (6.9) which, apart from (6.7), satisfy the following relations

$$F(\bar{X}, \bar{Y}) = 0, \qquad [\bar{X}, \bar{Y}] = 0 \quad \text{and} \quad \bar{\Delta} = \mathrm{Id}_n + \bar{\mathrm{v}}\,\bar{\mathrm{w}}.$$

The dual representation $\varrho^* : \Pi^\circ \to \mathrm{End}(\boldsymbol{V}^*)$ is given by the transposed matrices.

Now, (6.4) together with (6.8) show that $\kappa = \kappa(z, 1) \in Q$ is given by

$$\kappa = 1 + \bar{\mathrm{v}}^t\,(\bar{Z}^t - z\,\mathrm{Id}_n)^{-1}(\bar{X}^t - x\,\mathrm{Id}_n)^{-1}(\bar{Y}^t - y\,\mathrm{Id}_n)^{-1}F(\bar{X}^t, y\,\mathrm{Id}_n)\,\bar{\mathrm{w}}^t. \tag{6.11}$$

Thus, if $[\boldsymbol{V}] \in \mathcal{C}_n(X, A)$ is determined by the data $(\bar{X}, \bar{Y}, \bar{Z}, \bar{\mathrm{v}}, \bar{\mathrm{w}})$, then the class $\omega[\boldsymbol{V}]$ is represented by the (fractional) ideal

$$M = \mathcal{D} \cdot \det(\bar{X} - x\,\mathrm{Id}_n) + \mathcal{D} \cdot \det(\bar{Y} - y\,\mathrm{Id}_n) + \mathcal{D} \cdot \det(\bar{Z} - z\,\mathrm{Id}_n)\,\kappa.$$

In the general case, when $\mathcal{I}$ is arbitrary, κ is replaced by

$$\kappa(\mathrm{v}) = \mathrm{v} + \sum_i \left(\bar{\mathrm{v}}^t(\bar{Z}^t - z\,\mathrm{Id}_n)^{-1}(\bar{X}^t - x\,\mathrm{Id}_n)^{-1}(\bar{Y}^t - y\,\mathrm{Id}_n)^{-1}F(\bar{X}^t, y\,\mathrm{Id}_n)\,\bar{\mathrm{w}}_i^t\right)\mathrm{v}_i\,, \tag{6.12}$$

and the corresponding class $\omega[\boldsymbol{V}] \in \gamma^{-1}[\mathcal{I}]$ is given by

$$M = \sum_i [\mathcal{D} \cdot \det(\bar{X} - x\,\mathrm{Id}_n)\,\mathrm{v}_i + \mathcal{D} \cdot \det(\bar{Y} - y\,\mathrm{Id}_n)\,\mathrm{v}_i + \mathcal{D} \cdot \det(\bar{Z} - z\,\mathrm{Id}_n)\,\kappa(\mathrm{v}_i)]. \tag{6.13}$$

6.2.4. *A hyperelliptic curve*

This is a special plane curve described by the equation $y^2 = P(x)$, where $P(x) = \sum_s a_s x^s$ is a polynomial with simple roots. Some of the above formulae simplify in this case. We have $A \cong \mathbb{C}[x, y]/\langle y^2 - P(x)\rangle$, $\mathrm{Der}(A)$ is freely generated by ∂, with $\partial(x) = 2y$ and $\partial(y) = P'(x)$, and the bimodule $\mathbb{D}\mathrm{er}(A)$ is generated by Δ and the element z defined by

$$z(x) = y \otimes 1 + 1 \otimes y, \qquad z(y) = (P(x) \otimes 1 - 1 \otimes P(x))/(x \otimes 1 - 1 \otimes x).$$

The commutation relations (6.7) in $\mathbb{D}\mathrm{er}(A)$ are

$$[z, x] = y\Delta + \Delta y, \qquad [z, y] = \sum_s a_s \sum_{l=0}^{s-1} x^{s-l-1}\Delta x^l. \tag{6.14}$$

Now, for a hyperelliptic curve, a point of $\mathcal{C}_n(X, \mathcal{I})$ is determined by the following data: (1) a representation of A on the vector space $V = \mathbb{C}^n$, i.e. a pair of commuting matrices $(\bar{X}, \bar{Y}) \in \mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n)$ satisfying $\bar{Y}^2 = P(\bar{X})$; (2) a pair of A-module maps $\mathcal{I} \to V$ and $\mathcal{I}^\vee \to V^*$, with chosen images $\bar{\mathrm{v}}_i \in V$ and $\bar{\mathrm{w}}_i \in V^*$ of dual bases of $\mathcal{I}$ and $\mathcal{I}^\vee$; (3) a matrix $\bar{Z} \in \mathrm{End}(\mathbb{C}^n)$, such that $\bar{X}, \bar{Y}, \bar{Z}$ and $\bar{\Delta} := \mathrm{Id}_n + \sum_i \bar{\mathrm{v}}_i\bar{\mathrm{w}}_i$ satisfy (6.14). In this case, formula (6.12) reads

$$\kappa(\mathrm{v}) = \mathrm{v} - \sum_i \left(\bar{\mathrm{v}}^t(\bar{Z}^t - z\,\mathrm{Id})^{-1}(\bar{X}^t - x\,\mathrm{Id})^{-1}(\bar{Y}^t + y\,\mathrm{Id})\,\bar{\mathrm{w}}_i^t\right)\mathrm{v}_i,$$

and the corresponding ideal is given by (6.13).

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Appendix. Half-forms on Riemann surfaces

George Wilson

In this note I provide a proof for one of the key facts (Proposition A.1 below) needed to understand the relationship between deformed preprojective algebras and rings of differential operators. The note owes a great deal to conversations with Graeme Segal.

Statement of problem

Let X be a compact Riemann surface, and let Δ be the diagonal divisor in $X \times X$. We have the inclusion

$$\mathcal{O}_{X\times X}(-\Delta) \hookrightarrow \mathcal{O}_{X\times X}(\Delta)$$

of the sheaf of functions that vanish on Δ into the sheaf of functions that are allowed a simple pole on Δ. The quotient sheaf $\mathcal{O}_{X\times X}(\Delta)/\mathcal{O}_{X\times X}(-\Delta)$ is supported on the first infinitesimal neighbourhood Δ_1 of Δ. If $\mathcal{L}$ is a line bundle on X, we have the sheaf $\mathcal{D}_1(\mathcal{L})$ of differential operators of order ≤ 1 on $\mathcal{L}$. This is usually regarded as a sheaf on X, but since we can compose a differential operator with a function either on the left or on the right, it has two commuting structures of $\mathcal{O}_X$-module, so it too can be regarded as a sheaf on $X \times X$, again supported on Δ_1.

Fix a square root $\Omega^{1/2}$ of the canonical bundle Ω_X; the choice of square root will be immaterial, because the corresponding sheaves of differential operators $\mathcal{D}(\Omega^{1/2})$ are canonically isomorphic to each other. Our aim is to understand the following fact stated in [34].

Proposition A.1. *There is a canonical isomorphism (of sheaves over $X \times X$)*

$$\chi : \mathcal{O}_{X\times X}(\Delta)/\mathcal{O}_{X\times X}(-\Delta) \to \mathcal{D}_1(\Omega^{1/2}).$$

A consequence is that the sheaf of deformed preprojective algebras formed from $\mathcal{O}_X$ is canonically isomorphic to the sheaf $\mathcal{D}(\Omega^{1/2})$ of differential operators on $\Omega^{1/2}$. This is explained in [34], Section 13.

The isomorphism in Proposition A.1 does not seem to be a well-known fact, and at first sight looks puzzling, because there are no half-forms in the left hand side. It seems worth recording the following simple explanation shown to me by Segal: although Proposition A.1 itself does not look familiar, it can be obtained by combining two familiar facts, of a slightly different nature. While we are about it, we shall deal also with a slight generalization, twisting by an arbitrary line bundle $\mathcal{L}$ on X.

We use the following notation: Δ_n is the nth infinitesimal neighbourhood of the diagonal in $X \times X$, so that we have a canonical identification

$$\mathcal{L} \,/\, \mathcal{L}(-(n+1)\Delta) \;\simeq\; \mathcal{L} \mid \Delta_n. \tag{A.1}$$

The two projections $X \times X \to X$ are denoted by p_1 and p_2. If U is a simply-connected coordinate patch on X and z is a parameter on U, we write (z_1, z_2) for the induced parameters on $U \times U \subset X \times X$. The parameter z determines a trivialization (non-vanishing section) dz of $\Omega_X \mid U$. Fixing also an isomorphism[3] $\kappa : (\Omega^{1/2})^{\otimes 2} \simeq \Omega_X$, we may choose a trivialization $dz^{1/2}$ of $\Omega^{1/2} \mid U$ such that $\kappa(dz^{1/2} \otimes dz^{1/2}) = dz$ (there are only two choices, differing by a sign).

A proof of Proposition A.1

We shall use the following description of differential operators.

Proposition A.2. *Let $\mathcal{L}$ be a line bundle on X. Then there is a canonical identification (of sheaves over $X \times X$)*

$$p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^* \otimes \Omega_X)\,((n+1)\Delta) \mid \Delta_n \;\simeq\; \mathcal{D}_n(\mathcal{L}). \tag{A.2}$$

[3] Of course κ is uniquely determined up to a constant multiple. The isomorphism χ in Proposition A.1 does not depend on this multiple, but some of the intermediate steps below do.

Proof. The action of a (local) section of the sheaf on the left of (A.2) on a section of $\mathcal{L}$ is given by contracting with the factor $p_2^*(\mathcal{L}^*)$ and then taking the residue on the diagonal of the resulting differential. Let us spell that out in more detail in the case where $\mathcal{L}$ is the trivial bundle and $n = 1$. The sheaf on the left of (A.2) is then just $p_2^*(\Omega_X)(2\Delta) \mid \Delta_1 = p_2^*(\Omega_X)(2\Delta) / p_2^*(\Omega_X)$. In terms of a parameter z, a local section of this sheaf has the form

$$\frac{\varphi(z_1, z_2)\, dz_2}{(z_2 - z_1)^2} \quad \text{modulo regular terms}$$

(where φ is regular). To see how this acts on a function $f(z)$, we have to calculate the residue

$$\mathrm{res}_{z_2=z_1} \frac{f(z_2)\varphi(z_1, z_2) dz_2}{(z_2 - z_1)^2}$$

(z_1 is held fixed during the calculation). Expanding

$$f(z_2) = f(z_1) + f'(z_1)(z_2 - z_1) + \cdots,$$

and

$$\frac{\varphi(z_1, z_2)}{(z_2 - z_1)^2} = \frac{a(z_1)}{(z_2 - z_1)^2} + \frac{b(z_1)}{z_2 - z_1} + \cdots,$$

we find that the residue is

$$a(z)\frac{df}{dz} + b(z)f\Big|_{z=z_1}.$$

The proposition is now clear. □

Remark. As the proof shows, Proposition A.2 is just a coordinate-free formulation of Cauchy's formula expressing the derivatives $f^{(n)}(z_1)$ as contour integrals (or residues): see [23], p. 60, formule (4).

Now let U be a coordinate patch on X. We consider the classical[4] differential γ given in terms of a parameter z by

$$\gamma := \frac{dz_1^{1/2} dz_2^{1/2}}{z_1 - z_2}. \tag{A.3}$$

It is a non-vanishing section (over $U \times U$) of the line bundle

$$p_1^*(\Omega^{1/2}) \otimes p_2^*(\Omega^{1/2})(\Delta).$$

It depends on the parameter z; however, its restriction to Δ does not. Indeed, when we identify $\mathcal{O}_{X\times X}(-\Delta) \mid \Delta$ with the canonical bundle on the diagonal, $z_1 - z_2$ corresponds to dz, so $\gamma \mid \Delta$ becomes the constant section $1 \in \mathcal{O}(U)$. Furthermore, because γ is skew in the two variables, its restriction to Δ_1 is also independent of the choice of z. Thus for any sheaf $\mathcal{M}$ over $X \times X$, multiplication by γ gives a well-defined global isomorphism

$$\mathcal{M} \mid \Delta_1 \simeq \mathcal{M} \otimes p_1^*(\Omega^{1/2}) \otimes p_2^*(\Omega^{1/2})(\Delta) \mid \Delta_1.$$

In particular, for any line bundle $\mathcal{L}$ over X, we get an isomorphism

$$p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^*)(\Delta) \mid \Delta_1 \simeq p_1^*(\mathcal{L} \otimes \Omega^{1/2}) \otimes p_2^*(\mathcal{L}^* \otimes \Omega^{1/2})(2\Delta) \mid \Delta_1. \tag{A.4}$$

Tensoring our chosen isomorphism $\kappa : (\Omega^{1/2})^{\otimes 2} \simeq \Omega_X$ with $(\Omega^{1/2})^*$, we get an isomorphism $\Omega^{1/2} \simeq (\Omega^{1/2})^* \otimes \Omega_X$, and hence for any $\mathcal{L}$ an isomorphism

$$\mathcal{L}^* \otimes \Omega^{1/2} \simeq (\mathcal{L} \otimes \Omega^{1/2})^* \otimes \Omega_X.$$

Inserting this into (A.4) and taking account of (A.1) and (A.2) gives us an isomorphism (now independent of κ)

$$p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^*)(\Delta) / p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^*)(-\Delta) \simeq \mathcal{D}_1(\mathcal{L} \otimes \Omega^{1/2}). \tag{A.5}$$

Taking $\mathcal{L} = \mathcal{O}_X$, we get Proposition A.1.

[4] It is the principal part of the *Szegö kernel* on $X \times X$.

Remarks

1. Reversing the arguments in [34], we easily get from (A.5) a construction of any $\mathcal{D}(\mathcal{L})$ as a sheaf of 'twisted deformed preprojective algebras'.
2. The differential γ in (A.3) is invariant under a *linear fractional* change of parameter. Thus if we fix a projective structure on X (thought of as an atlas with linear fractional transition functions), then γ is well-defined globally on some analytic neighbourhood of Δ, not merely on Δ_1. This remark is the starting point for the papers [17].
3. The considerations above give an explicit formula for the isomorphism χ in Proposition A.1: an element of $\mathcal{O}_{X\times X}(\Delta)/\mathcal{O}_{X\times X}(-\Delta)$ has a unique local representative of the form

$$a(z_1)(z_2 - z_1)^{-1} + b(z_1) + \cdots ,$$

and χ maps this to the operator

$$f\, dz^{1/2} \mapsto \left(a(z)\frac{df}{dz} + b(z)f\right) dz^{1/2}.$$

In an earlier version of this note I verified Proposition A.1 by checking directly that the map χ defined by this formula is independent of the chosen parameter z; however, the calculation is surprisingly complicated (and unilluminating).

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